

Some Applications of Extended Interval Arithmetic to Interval Iterations

S. M. Markov, Sofia

Abstract

The calculation of united extensions of real functions of one variable by means of primitive interval operations is considered. It is demonstrated that extended interval arithmetic is a convenient tool for treating this problem. Some direct applications of the results obtained to interval iteration procedures are given.

1. Introduction

Familiar interval mathematics makes use of only two primitive arithmetic operations between intervals: addition $A + B$ and multiplication AB of the intervals A and B . The operations for the familiar subtraction and division are compositions of these two operations. Indeed, the subtraction of $A = [a_1, a_2]$, $B = [b_1, b_2]$ can be written as $A + [-1, -1]B$, and division is $[c, c]AB$, where $c = 1/(b_1 b_2)$.

Therefore we should not consider the last two operations as primitive ones. We shall call them auxiliary subtraction and auxiliary division, respectively, in order to avoid confusion with other operations that are introduced later.

As it is based upon only two primitive operations, familiar interval arithmetic presents a very simple algebraic structure. It is often realized that this algebraic structure is not rich enough to handle various problems arising in interval mathematics. This structure can be substantially enriched if the set of primitive operations is extended by two new operations. This extended interval arithmetic presents an interesting algebraical structure that is especially suitable for handling problems involving interval functions.

In this paper we give a short introduction to extended interval arithmetic and some rules for calculating united extensions of real functions of one variable. We then apply these results to some interval iteration procedures of Newton type.

2. Extended Interval Arithmetic

We shall denote the set of reals by \mathcal{R} and the set of all closed intervals on \mathcal{R} by $\mathcal{I}(\mathcal{R})$. Let a_1 and a_2 , $a_1 \leq a_2$, be the end-points of $A \in \mathcal{I}(\mathcal{R})$, and write $A = [a_1, a_2]$. By a_c and a_d we mean the end-points of A such that $|a_c| \leq |a_d|$; that is, a_c is the end-point of A that is closer to zero than a_d . An interval with end-points α and β (α not necessarily

$\leq \beta$) will be denoted by $\alpha \vee \beta$ or $[\alpha \vee \beta]$. For the width of A we write $w(A) = a_2 - a_1$. The absolute value of A is $|A| = \max\{|a_1|, |a_2|\}$. The set of all intervals A , such that $0 \in A$, will be denoted by $\mathcal{I}^*(\mathcal{A})$.

Addition of $A, B \in \mathcal{I}(\mathcal{A})$ is defined by

$$(A) \quad A + B = [a_1 + b_1, a_2 + b_2].$$

A convenient formula for multiplication of two intervals $A, B \in \mathcal{I}^*(\mathcal{A})$ is

$$(M) \quad AB = (a_c b_c) \vee (a_d b_d).$$

Auxiliary subtraction $A + (-B)$ will be indicated by $A \ominus B$, and this may be written as

$$(AS) \quad A \ominus B = [a_1 - b_2, a_2 - b_1],$$

for $A, B \in \mathcal{I}(\mathcal{A})$.

Auxiliary division of A and B for $A, B \in \mathcal{I}^*(\mathcal{A})$ can be written as

$$(AD) \quad A \odot B = (a_c/b_d) \vee (a_d/b_c).$$

The operations (A), (M), (AS), and (AD) are well known from familiar interval arithmetic [1, 5, 6, etc.], since we have

$$A + B = \{a + b : a \in A, b \in B\}, \quad A \ominus B = \{a - b : a \in A, b \in B\},$$

for $A, B \in \mathcal{I}(\mathcal{A})$ and

$$AB = \{ab : a \in A, b \in B\}, \quad A \odot B = \{a/b : a \in A, b \in B\},$$

for $A, B \in \mathcal{I}^*(\mathcal{A})$.

We extend [3] the interval arithmetic by the following two operations: Basic subtraction, given for $A, B \in \mathcal{I}(\mathcal{A})$ by

$$(S) \quad A - B = (a_1 - b_1) \vee (a_2 - b_2),$$

and basic division, defined for $A, B \in \mathcal{I}^*(\mathcal{A})$ by

$$(D) \quad A/B = (a_c/b_c) \vee (a_d/b_d).$$

We shall further consider the operations (A), (M), (S), and (D) as primitive operations. The auxiliary operations (AS) and (AD) are compositions of the primitive operations, since $A \ominus B = A + (-B)$ and $A \odot B = A(1/B)$. Analogously, we may consider an auxiliary addition $A \oplus B = A - (-B)$ and an auxiliary multiplication $A \otimes B = A/(1/B)$.

These operations can be expressed in terms of end-points of the intervals as follows:

$$(AA) \quad A \oplus B = (a_1 + b_2) \vee (a_2 + b_1) \quad \text{for } A, B \in \mathcal{I}(\mathcal{A}),$$

$$(AM) \quad A \otimes B = (a_c b_d) \vee (a_d b_c) \quad \text{for } A, B \in \mathcal{I}^*(\mathcal{A}).$$

Let us give expressions for AB , A/B , $A \otimes B$, and $A \odot B$ for intervals containing zero. In the usual interval arithmetic the operations AB and $A \odot B$ are defined by

$AB = \{ab : a \in A, b \in B\}$ for arbitrary A, B and $A \odot B = \{a/b, a \in A, b \in B\}$ for $0 \in B$ and arbitrary A . By means of the end-points these definitions can be written:

$$\left. \begin{aligned} AB &= b_d A = (a_c b_d) \vee (a_d b_d) \\ A \odot B &= (1/b_c)A = (a_c/b_c) \vee (a_d/b_c) \end{aligned} \right\} \text{ for } 0 \in A, 0 \in B,$$

and

$$AB = [\min\{a_1 b_2, a_2 b_1\}, \max\{a_1 b_1, a_2 b_2\}] \quad \text{for } 0 \in A, 0 \in B.$$

Thus we may define $A \otimes B$ and A/B for intervals containing zero by

$$\left. \begin{aligned} A \otimes B &= b_c A = (a_c b_c) \vee (a_d b_c) \\ A/B &= (1/b_d)A = (a_c/b_d) \vee (a_d/b_d) \end{aligned} \right\} \text{ for } 0 \in A, 0 \in B$$

and

$$A \otimes B = [\max\{a_1 b_2, a_2 b_1\}, \min\{a_1 b_1, a_2 b_2\}] \quad \text{for } 0 \in A, 0 \in B.$$

Then we have $A \otimes B = A/(1/B)$ and $A \odot B = A(1/B)$ for all $A \in \mathcal{I}(\mathcal{A})$ and $B \in \mathcal{I}^*(\mathcal{A})$.

Table 1 summarizes the definitions of the basic and primitive operations used in extended interval arithmetic.

Table 1

	Basic Operations	Auxiliary Operations
$A, B \in \mathcal{I}(\mathcal{A})$	$A + B = (a_1 + b_1) \vee (a_2 + b_2)$ $A - B = (a_1 - b_1) \vee (a_2 - b_2)$	$A \oplus B = (a_1 + b_2) \vee (a_2 + b_1) = A - (-B)$ $A \ominus B = (a_1 - b_2) \vee (a_2 - b_1) = A + (-B)$
$A, B \in \mathcal{I}^*(\mathcal{A})$	$AB = (a_c b_c) \vee (a_d b_d)$ $A/B = (a_c/b_c) \vee (a_d/b_d)$	$A \otimes B = (a_c b_d) \vee (a_d b_c) = A/(1/B)$ $A \odot B = (a_c/b_d) \vee (a_d/b_c) = A(1/B)$
$0 \in A$	$AB = b_d A$	$A \otimes B = b_c A = A/(1/B)$
$0 \in B$	$A/B = (1/b_d)A$	$A \odot B = (1/b_c)A = A(1/B)$
$0 \in A$	$AB = [\min\{a_1 b_2, a_2 b_1\},$	$A \otimes B = [\max\{a_1 b_2, a_2 b_1\},$
$0 \in B$	$\max\{a_1 b_1, a_2 b_2\}]$	$\min\{a_1 b_1, a_2 b_2\}]$

3. Interval Operators and Fixed Points

Let \mathcal{L} be a normed lattice and $\mathcal{I}(\mathcal{L})$ be the corresponding normed interval space over \mathcal{L} , see [4]. Denote by $\|\cdot\|$ the norm in $\mathcal{I}(\mathcal{L})$ and consider an interval operator $U: \mathcal{I}(\mathcal{L}) \rightarrow \mathcal{I}(\mathcal{L})$. The operator U is a contraction mapping in $\mathcal{I}(\mathcal{L})$ if there is a constant $q, 0 < q < 1$, such that the inequality

$$\|U(X) - U(Y)\| \leq q\|X - Y\|$$

holds for every $X, Y \in \mathcal{I}(\mathcal{L})$.

Assume further that \mathcal{L} is a Banach lattice and hence is complete. Then we have the following fixed-point theorem (as usual a fixed point of U is an $X^* \in \mathcal{I}(\mathcal{L})$ such that $X^* = U(X^*)$):

Theorem 1. *Let \mathcal{L} be a Banach lattice and U be a contraction mapping in $\mathcal{I}(\mathcal{L})$. Then the operator U possesses a unique fixed point $X^* \in \mathcal{I}(\mathcal{L})$, that is the limit of the sequence of successive approximations*

$$X^{(n+1)} = U(X^{(n)}), \quad n = 0, 1, 2, \dots,$$

with $X^{(0)} \in \mathcal{I}(\mathcal{L})$ arbitrary.

Proof. The operation “ $-$ ” in $\mathcal{I}(\mathcal{L})$ enables us to prove Theorem 1 in exactly the same way as in the classical case. Indeed, using the equality $X^{(n+1)} = U(X^{(n)})$ and the properties of the interval norm $\|\cdot\|$, we have for $m > n > 0$:

$$\begin{aligned} \|X^{(n)} - X^{(m)}\| &= \|U(X^{(n-1)}) - U(X^{(m-1)})\| \leq q \|X^{(n-1)} - X^{(m-1)}\| \\ &= q \|U(X^{(n-2)}) - U(X^{(m-2)})\| \leq q \|X^{(n-2)} - X^{(m-2)}\| \leq \dots \\ &\leq q^n \|X^{(0)} - X^{(m-n)}\| \dots \end{aligned}$$

On the other side, we have

$$\begin{aligned} \|X^{(0)} - X^{(m-n)}\| &\leq \|X^{(0)} - X^{(1)}\| + \|X^{(1)} - X^{(2)}\| + \dots + \|X^{(m-n-1)} - X^{(m-n)}\| \\ &\leq (1 + q + q^2 + \dots + q^{m-n-1}) \|X^{(0)} - X^{(1)}\| \\ &\leq (1 - q)^{-1} \|X^{(0)} - X^{(1)}\|. \end{aligned}$$

Thus

$$\|X^{(n)} - X^{(m)}\| \leq q^n (1 - q)^{-1} \|X^{(0)} - X^{(1)}\|,$$

and therefore $X^{(n)}$ is an interval Cauchy sequence, that is $\lim_{n \rightarrow \infty} \|X^{(n)} - X^{(m)}\| = 0$. From the completeness of \mathcal{L} it follows then that there exists an X^* , such that

$$\lim_{n \rightarrow \infty} \|X^{(n)} - X^*\| = 0.$$

This result and $\|U(X^{(n)}) - U(X^*)\| \leq q \|X^{(n)} - X^*\|$ imply $\lim_{n \rightarrow \infty} \|U(X^{(n)}) - U(X^*)\| = 0$ as well. Further, from

$$\begin{aligned} \|X^* - U(X^*)\| &\leq \|X^* - X^{(n)}\| + \|X^{(n)} - U(X^*)\| \\ &= \|X^* - X^{(n)}\| + \|U(X^{(n-1)}) - U(X^*)\| \rightarrow 0 \end{aligned}$$

we see that $\|X^* - U(X^*)\| = 0$ and hence $X^* = U(X^*)$. This completes the proof.

We next formulate the fixed-point theorem for the case when U is a contraction mapping only in a neighbourhood of the fixed point.

Theorem 2. *Let \mathcal{L} be a Banach lattice and U be a contraction mapping in a neighbourhood $\mathcal{V} \subset \mathcal{I}(\mathcal{L})$ of its fixed point X^* . Then the iteration process $X^{(n+1)} = U(X^{(n)})$, $n = 0, 1, \dots$, converges to X^* as $O(q^n)$ for any $X^{(0)} \in \mathcal{V}$.*

Proof. The assumptions on U mean that there exist two positive numbers p and $q < 1$, such that $\|U(X) - U(Y)\| \leq q \|X - Y\|$ for every two points $X, Y \in \mathcal{V}(X^*, p) = \{Z: \|Z - X^*\| < p, Z \in \mathcal{I}(\mathcal{L})\}$.

Assume that $X^{(m)} \in \mathcal{V}(X^*, p)$ for some $m = 1, 2, \dots$. Then

$$\|X^{(m+1)} - X^*\| = \|U(X^{(m)}) - X^*\| \leq q \|X^{(m)} - X^*\| < qp < p$$

shows that $X^{(m+1)} \in U(X^*, p)$ as well. Since $X^{(0)} \in U(X^*, p)$, we obtain that $X^{(i)} \in U(X^*, p)$ for all $i = 1, 2, \dots$. Using the fact that U is a contraction mapping in $U(X^*, p)$ we can write

$$\begin{aligned} \|X^{(m)} - X^*\| &= \|U(X^{(n-1)}) - U(X^*)\| \leq q \|X^{(n-1)} - X^*\| \leq \dots \\ &\leq q^n \|X^{(0)} - X^*\| \leq pq^n, \end{aligned}$$

and this proves the theorem.

As usual we say that when $\|X^{(n)} - X^*\| = O(q^n)$, the convergence is linear or of order 1. More generally, the convergence is called of order $p > 1$ if $\|X^{(n)} - X^*\| = O(q^{pn})$. Thus in the above we have linear convergence; later we shall consider interval iteration procedures with convergence of order 2.

4. United Extensions of Real Functions and Their Computation by Means of Interval Arithmetic Operations

Let f be a real function defined on the interval D and $\mathcal{I}(D)$ be the set of all subintervals of D . A mapping $F^*: \mathcal{I}(D) \rightarrow \mathcal{I}(\mathcal{R})$ is called an interval extension of f , if the restriction of F^* to D is equal to f , $F^*|_D = f$. A special case of interval extension is the united extension, defined by

$$F(X) = \bigvee \{f(x) : x \in X\}, \quad \text{for } X \in \mathcal{I}(D). \quad (1)$$

Here $\bigvee \{f(x) : x \in X\}$ means the interval of minimal width, containing all $f(x)$ for $x \in X$.

We shall denote the united extension of the functions f, g, \dots by F, G, \dots , respectively.

If f is continuous, then (1) can be written

$$F(X) = \bigvee \{f(x) : x \in X\} = [\min_{x \in X} f(x), \max_{x \in X} f(x)].$$

We shall now consider the following problem: How do we calculate the united extensions of the functions $f + g$, $f - g$, f/g , and $f \cdot g$ if the united extensions of the functions f and g are known? We shall answer this question in case that f and g satisfy certain monotonicity conditions.

Denote by $\mathcal{M}(D)$ the set of all real functions that are monotone on the interval D . Note that the united extension of $f \in \mathcal{M}(D)$ is easily computed by

$$F(X) = [f(x_1) \vee f(x_2)] \quad \text{for } X = [x_1, x_2] \in \mathcal{I}(D). \quad (2)$$

We shall say that $f, g \in \mathcal{M}(D)$ satisfy the monotonicity condition $\mathcal{M}1$ if both functions are monotone increasing or both are monotone decreasing in D . The pair (f, g) satisfies $\mathcal{M}2$ if one of the functions is monotone increasing and the other is monotone decreasing on D .

Proposition 1. If f, g and $h = f + g \in \mathcal{M}(D)$, then for every $X \in \mathcal{I}(D)$:

$$H(X) = \begin{cases} F(X) + G(X), & \text{if } (f, g) \text{ satisfy } \mathcal{M}1, \\ (F(X) \oplus G(X)), & \text{if } (f, g) \text{ satisfy } \mathcal{M}2. \end{cases} \quad (3)$$

Proposition 2. If f, g , and $h = f - g$ are monotone on D , then for every $X \subset D$:

$$H(X) = \begin{cases} F(X) - G(X), & \text{if } (f, g) \text{ satisfy } \#1, \\ F(X) \ominus G(X), & \text{if } (f, g) \text{ satisfy } \#2. \end{cases} \quad (4)$$

Proposition 3. Let f and g are such that $|f|, |g|$, and $h = f \cdot g$ are monotone. Then for every $X \subset D$,

$$H(X) = \begin{cases} F(X) \cdot G(X), & \text{if } (|f|, |g|) \text{ satisfy } \#1, \\ F(X) \otimes G(X), & \text{if } (|f|, |g|) \text{ satisfy } \#2. \end{cases} \quad (5)$$

Proposition 4. Let f and g are such that $|f|, |g| \in \#(D)$, $g(x) \neq 0$ for $x \in D$, and $h = f/g \in \#(D)$. Then for every $X \subset D$,

$$H(X) = \begin{cases} F(X) : G(X), & \text{if } (|f|, |g|) \text{ satisfy } \#1, \\ F(X) \odot G(X), & \text{if } (|f|, |g|) \text{ satisfy } \#2. \end{cases} \quad (6)$$

The verification of these four propositions is straightforward. As an example we prove the last one.

Verification of Proposition 4. Using (2) for $f/g \in \#(D)$ we can write $H(X) = [(f(x_1)/g(x_1)) \vee (f(x_2)/g(x_2))]$. Now if $|f|$ and $|g|$ satisfy $\#1$, then $(|f(x_1)| - |f(x_2)|)(|g(x_1)| - |g(x_2)|) \geq 0$. In this case we see easily that

$$\begin{aligned} H(X) &= [(f(x_1)/g(x_1)) \vee (f(x_2)/g(x_2))] \\ &= [f(x_1) \vee f(x_2)] : [g(x_1) \vee g(x_2)] = F(X) : G(X). \end{aligned}$$

In case $|f|$ and $|g|$ satisfy $\#2$, we have that

$$(|f(x_1)| - |f(x_2)|)(|g(x_1)| - |g(x_2)|) < 0$$

implies

$$\begin{aligned} H(X) &= [(f(x_1)/g(x_1)) \vee (f(x_2)/g(x_2))] \\ &= [f(x_1) \vee f(x_2)] \odot [g(x_1) \vee g(x_2)] = F(X) \odot G(X). \end{aligned}$$

Let us now give some practical applications of Propositions 1–4.

Example 1. Find the united extension of the function $h(x) = x^2 + x$, for $x \in \mathcal{R}$.

Solution. The united extension of x is X . By means of Proposition 3 we see that the united extension of $g(x) = x^2 = x \cdot x$ can be calculated for intervals $X \bar{\ni} 0$ (since for such intervals the pair $(|x|, |x|)$ satisfies $\#1$) and is equal to $X \cdot X = X^2$. Applying Proposition 1 we see that the interval extension of $x + x^2$ can be calculated for every interval X that does not contain the points -0.5 and zero in its interior since on such X the functions x, x^2 and $x + x^2$ are monotone. It is given by

$$H(X) = \{x + x^2 : x \in X\} = \begin{cases} X + X^2, & \text{if } X \geq 0; \\ X \oplus X^2, & \text{if } X \leq -0.5 \text{ or } X \subset [-0.5, 0]. \end{cases}$$

Example 2. For the united extension of the function $h(x) = x - x^2$, we find upon applying Proposition 2, that

$$H(X) = \begin{cases} X - X^2, & \text{if } X \in [0, +0.5] \text{ or } X \geq 0.5; \\ X \ominus X^2, & \text{if } X \leq 0. \end{cases}$$

Example 3. Find the united extension of the rational function $h(x) = (2 + x)/(1 + x)$.

Solution. The functions $|2 + x|$ and $|1 + x|$ satisfy #1 in the intervals $(-x, -2)$ and $[-1, x]$ and satisfy #2 in $[-2, -1]$. We also have $h \in \#((-x, -1])$ and $h \in \#([-1, x])$. Therefore, in accord with Proposition 4 we may write

$$H(X) = \begin{cases} (2 + X)/(1 + X), & \text{if } X \leq -2 \text{ or } X \geq -1; \\ (2 + X) \odot (1 + X), & \text{if } -1 \leq X \leq -2. \end{cases}$$

where we have also used the fact that the united extensions of the functions $2 + x$ and $1 + x$ are $2 + X$ and $1 + X$, respectively. (This example is taken from [2], where it is regretfully noted that the united extension of $(2 + x)/(1 + x)$ cannot be calculated by means of the familiar primitive interval arithmetic operations for $X \in [1, 2]$).

These examples show that it is possible to outline an algorithm for calculating the united extensions of an arbitrary rational function of one variable in certain intervals.

5. United Extensions of Some Newton-Type Operators

We shall say that f satisfies the N -condition on $D = [d_1, d_2]$ if:

- 1) there is an x^* , $d_1 < x^* < d_2$, such that $f(x^*) = 0$;
- 2) f is twice continuously differentiable in D ;
- 3) f' and f'' have constant sign in D ;
- 4) f/f' is monotone increasing in D ;
- 5) f'' is not very close to zero in D , i.e. $|f''| \geq \lambda > 0$ in D .

Denote by D' the subinterval of D on which $f(x)f''(x) \geq 0$; D' is either the interval $[d_1, x^*]$ or the interval $[x^*, d_2]$. Similarly let D'' be the other subinterval of D on which $f(x)f''(x) \geq 0$.

Define the Newton operators

$$n(x) = n(f; x) = x - f(x)/f'(x)$$

$$\tilde{n}(x) = \tilde{n}(f; x) = x - f(x)/f'(\tilde{d}),$$

where \tilde{d} is such that $f'(\tilde{d}) = \max_{x \in D} |f'(x)|$. Obviously $\tilde{d} \in \{d_1, d_2\}$.

We shall first calculate the united extension of $\tilde{n}(x)$. To this end note that the functions $g(x) = f(x)/f'(\tilde{d})$ and $\tilde{n}(x) = x - g(x)$ are monotone increasing on D (indeed, $\tilde{n}'(x) = 1 - f'(x)/f'(\tilde{d}) \geq 0$ in D). Then by means of Proposition 2 we obtain for the united extension of $\tilde{n}(x)$:

$$\tilde{N}(X) = X - F(X)/f'(\tilde{d}).$$

We notice that \tilde{d} is an end-point of D (the one that is also end-point of D'). If we recall the definition of the operation A/B for the case $A \ni 0$ (see Table 1) we see that

$\tilde{N}(X)$ can be written as

$$\tilde{N}(X) = X - F(X)F'(D), \quad (7)$$

where F' is the united extension of f' .

We shall now calculate the united extension of $n(x)$. In order to write down the united extension G of $g(x) = f(x)/f''(x)$ we note that $|f|$ and $|f''|$ satisfy //1 on D' and //2 on D'' . Since f/f'' is monotone increasing in D we can apply Proposition 4 to obtain

$$G(X) = \begin{cases} F(X)F''(X) & \text{for } X \subset D', \\ F(X) \odot F''(X) & \text{for } X \subset D'', \end{cases}$$

where F and F'' are the united extensions of f and f'' , respectively.

Now in order to calculate the united extension of $n(x) = x - g(x)$ we first observe that x and $g(x)$ are both monotone increasing in D and that $n(x)$ is monotone in D and D' (since $n'(x) = f'(x)f''(x) - f''^2(x)$). Therefore by Proposition 2 we obtain $N(X) = X - G(X)$, for $X \subset D', D''$, that is

$$N(X) = \begin{cases} X - F(X)F''(X), & \text{for } X \text{ such that } F(X)F''(X) \geq 0, \\ X - F(X) \odot F''(X), & \text{for } X \text{ such that } F(X)F''(X) \leq 0, \end{cases} \quad (8)$$

where F , F' and F'' are the united extensions of f , f' and f'' , respectively.

According to (8) if X is an interval, such that $f'(x)f''(x) \geq 0$ for all $x \in X$, then

$$N(X) = \{x - f(x)/f''(x) : x \in X\} = X - F(X)F''(X),$$

where f satisfies the N -conditions on $D \subset X$ (see Fig. 1).

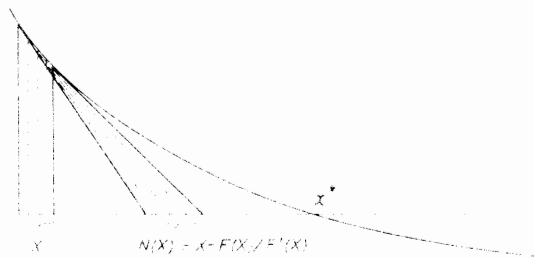


Fig. 1

Assume that the equation $f(x) = 0$ is solved by the iteration procedure $x^{(k+1)} = x^{(k)} - f(x^{(k)})/f'(x^{(k)})$ with $x^{(0)}$ such that $f'(x^{(0)})f''(x^{(0)}) > 0$. Suppose we know bounds for $x^{(0)}$, that is, suppose we know (small) interval $X^{(0)}$ containing $x^{(0)}$. Then we can assert that $x^{(1)} \in X^{(1)} = X^{(0)} - F(X^{(0)})F'(X^{(0)})$ and by induction $x^{(k+1)} \in X^{(k+1)} = X^{(k)} - F^{(k)}/F'(X^{(k)})$. Thus by means of extended interval arithmetic we can calculate an interval for $x^{(k+1)}$ provided we know an interval for $x^{(0)}$.

6. Some Interval Iteration Procedures of Newton Type

As in the previous section we shall assume that f satisfies the N -conditions on D . We shall demonstrate convergency of the interval iteration procedures $X^{(n+1)} = \tilde{N}(X^{(n)})$ and $X^{(n+1)} = N(X^{(n)})$ where \tilde{N} and N are the Newton-type interval operators defined in Sect. 5.

Theorem 3. *Let f satisfy the N -conditions on D and F be the united extension of f . Then the iteration scheme*

$$\begin{aligned} X^{(0)} &= D \\ X^{(i+1)} &= X^{(i)} - F(X^{(i)})/F'(X^{(i)}), \quad i = 0, 1, 2, \dots \end{aligned} \quad (9)$$

produces an interval sequence $\{X^{(i)}\}$ converging to x^ , that is $\lim_{i \rightarrow \infty} |X^{(i)} - x^*| = 0$.*

Proof. As we showed $\tilde{N}(X) = X - F(X)/F'(D)$ is the united extension of $\tilde{n}(x) = x - f(x)/f'(\tilde{d})$. We shall demonstrate that $\tilde{N}(X)$ satisfies $|\tilde{N}(X) - \tilde{N}(Y)| \leq q|X - Y|$ with $q < 1$ for every $X, Y \subset D$. Indeed, since $\tilde{N}(X)$ is the united extension of $\tilde{n}(x)$ and \tilde{n} is monotone increasing in D , we can write $\tilde{N}(X) = [\tilde{n}(x_1), \tilde{n}(x_2)]$. Further, using that $|\tilde{n}(u) - \tilde{n}(v)| \leq q|u - v|$, $q < 1$ for $u, v \in D$, we have

$$\begin{aligned} |\tilde{N}(X) - \tilde{N}(Y)| &= |[\tilde{n}(x_1), \tilde{n}(x_2)] - [\tilde{n}(y_1), \tilde{n}(y_2)]| \\ &= |[(\tilde{n}(x_1) - \tilde{n}(y_1)) \vee (\tilde{n}(x_2) - \tilde{n}(y_2))]| \\ &= \max\{|\tilde{n}(x_1) - \tilde{n}(y_1)|, |\tilde{n}(x_2) - \tilde{n}(y_2)|\} \\ &\leq \max\{q|x_1 - y_1|, q|x_2 - y_2|\} = q|X - Y|. \end{aligned}$$

Thus $\tilde{N}(X)$ satisfies the conditions of Theorem 2 with fix-point $X^* = x^*$. Therefore (9) produces a sequence $X^{(k)} \rightarrow x^*$.

Remark 1. Theorem 3 holds true for arbitrary chosen $X^{(0)} \subset D$ (not necessarily $X^{(0)} \ni x^*$).

Remark 2. Theorem 3 follows directly from the theorem in [2].

We shall next consider the interval iteration procedure using the operator $N(X)$.

Theorem 4. *The iteration scheme*

$$\begin{aligned} &\text{choose } X^{(0)} \subset D \\ &\text{let } X^{(i+1)} = X^{(i)} - F(X^{(i)})/F'(X^{(i)}) \end{aligned} \quad (10)$$

produces an interval sequence $\{X^{(i)}\}$ that converges to x^ and for the rate of convergence we have*

$$|X^{(i)} - x^*| = O(q^2), \quad 0 < q < 1.$$

Proof. We shall first consider the case $x^* \in X^{(0)}$. This case is the most interesting for the practice, since then by means of (10) we can obtain a sufficiently narrow interval that contains the zero x^* of $f(x) = 0$.

There are four subcases according to the signs of f' and f'' :

- a) $f' > 0$ and $f'' < 0$ (Fig. 2a);
- b) $f' < 0$ and $f'' > 0$ (Fig. 2b);
- c) $f' > 0$ and $f'' > 0$ (Fig. 2c);
- d) $f' < 0$ and $f'' < 0$ (Fig. 2d).

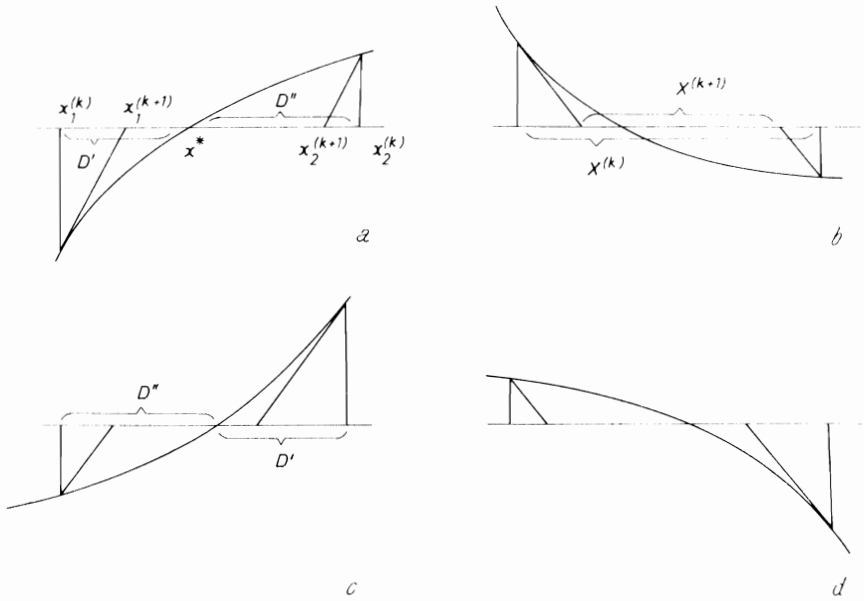


Fig. 2a-d

All subcases are treated similarly; we shall consider in detail the first subcase. $f' > 0$ and $f'' < 0$ imply (see Fig. 2a):

$$F(X^{(k)}) = [f(x_1^{(k)}), f(x_2^{(k)})],$$

$$F'(X^{(k)}) = [f'(x_2^{(k)}), f'(x_1^{(k)})] > 0.$$

Since $F(X^{(k)})$ contains zero and $F'(X^{(k)}) > 0$, we have $F(X^{(k)})/F'(X^{(k)}) = F(X^{(k)})/f'(x_1^{(k)})$. The function $x - f(x)/f'(x_1^{(k)})$ increases in $X^{(k)}$ so that we have

$$x_2^{(k)} - f(x_2^{(k)})/f'(x_1^{(k)}) \geq x_1^{(k)} - f(x_1^{(k)})/f'(x_1^{(k)}).$$

Therefore iteration (10) can be written end-point wise:

$$x_1^{(k+1)} = x_1^{(k)} - f(x_1^{(k)})/f'(x_1^{(k)})$$

$$x_2^{(k+1)} = x_2^{(k)} - f(x_2^{(k)})/f'(x_1^{(k)}) \quad (11)$$

(see Fig. 2a).

In subcase b) for the end-point formulation of (10) we obtain again (11), whereas in subcases c) and d) we obtain

$$\begin{aligned} X_1^{(k+1)} &= X_1^{(k)} - f(X_1^{(k)})/f'(X_2^{(k)}) \\ X_2^{(k+1)} &= X_2^{(k)} - f(X_2^{(k)})/f'(X_2^{(k)}) \end{aligned}$$

Summarizing, for (10) we have

$$X^{(k+1)} = X^{(k)} - F(X^{(k)})/F'(X^{(k)}) = X^{(k)} - F(X^{(k)})/f'(X_j^{(k)}), \quad (10')$$

where $j = 1$, if $f'f'' < 0$ in $X^{(0)}$ and $j = 2$, if $f'f'' > 0$ in $X^{(0)}$.

This shows that only one computation of f' on each step is required.

We consider now the convergence of the method.

In subcase a) we can write for the width of $F(X^{(k)})$:

$$\begin{aligned} w(F(X^{(k)})) &= f(X_2^{(k)}) - f(X_1^{(k)}) = (X_2^{(k)} - X_1^{(k)})f'(X_1^{(k)}) + \frac{1}{2}(X_2^{(k)} - X_1^{(k)})^2 f''(c^{(k)}) \\ &= w(X^{(k)})f'(X_1^{(k)}) + \frac{1}{2}w^2(X^{(k)})f''(c^{(k)}), \quad c^{(k)} \in X^{(k)} \end{aligned} \quad (12)$$

Since $f'' < 0$ (12) implies that

$$w(F(X^{(k)})/f'(X_1^{(k)})) < w(X^{(k)}). \quad (13)$$

From (10') using $w(A - B) = w(A) - w(B)$ and (13) we have

$$\begin{aligned} w(X^{(k+1)}) &= w((X^{(k)} - F(X^{(k)})/f'(X_1^{(k)})) \\ &= w(X^{(k)}) - w(F(X^{(k)})/f'(X_1^{(k)})). \end{aligned}$$

Substituting (12) in the above equality we obtain:

$$\begin{aligned} w(X^{(k+1)}) &= w(X^{(k)}) - (w(X^{(k)})f'(X_1^{(k)}) + \frac{1}{2}w^2(X^{(k)})f''(c^{(k)}))/f'(X_1^{(k)}) \\ &= - (f''(c^{(k)})/2f'(X_1^{(k)})) \cdot w^2(X^{(k)}), \quad c^{(k)} \in X^{(k)}. \end{aligned}$$

Subcases b), c), and d) can be treated in a similar way; the results obtained can be summarized as follows:

$$w(X^{(k+1)}) = -\frac{1}{2}(f''(c^{(k)})/f'(X_j^{(k)})) \cdot w^2(X^{(k)}),$$

where j is 1 or 2. Choosing $q > 0$ such that $-\frac{1}{2}(f''(c)/f'(d)) \leq q$ for all $c, d \in D$ we may write

$$w(X^{(k+1)}) \leq q \cdot w^2(X^{(k)}). \quad (14)$$

In view of $X^* \in X^{(k)}$ (14) can be written also as

$$|X^{(k+1)} - X^*| \leq 4q|X^{(k)} - X^*|^2 \quad (15)$$

Indeed, using the fact that $X^* \in X^{(k)}$ implies

$$\frac{1}{2}w(X^{(k)}) \leq |X^{(k)} - X^*| \leq w(X^{(k)}),$$

we have

$$|X^{(k+1)} - X^*| \leq w(X^{(k+1)}) \leq qw^2(X^{(k)}) \leq 4q(\frac{1}{2}w(X^{(k)}))^2 \leq 4q|X^{(k)} - X^*|^2.$$

Now from (15) and Theorem 2 the proof of Theorem 4 follows for the case $X^* \in X^{(0)}$.

If $x^* \in X^{(0)}$, then either $X^{(0)} \subset D'$ or $X^{(0)} \subset D''$. Consider the case $X^{(0)} \subset D'$. In this case (10) can be written in terms of the end-points

$$X^{(k+1)} = \left(x_1^{(k)} - \frac{f(x_1^{(k)})}{f'(x_1^{(k)})} \right) \vee \left(x_2^{(k)} - \frac{f(x_2^{(k)})}{f'(x_2^{(k)})} \right)$$

which is the usual Newton method starting one time from $x_1^{(k)}$ and another time from $x_2^{(k)}$ (Fig. 3).

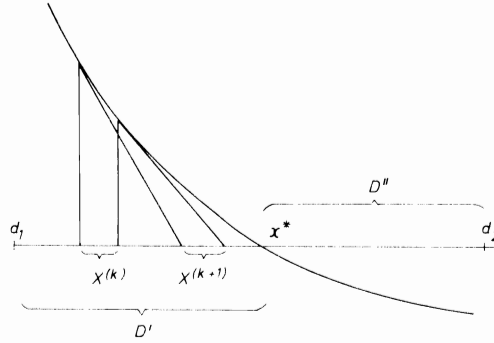


Fig. 3

From the well-known inequalities

$$|x_1^{(k+1)} - x^*| \leq q|x_1^{(k)} - x^*|^2, \quad |x_2^{(k+1)} - x^*| \leq q|x_2^{(k)} - x^*|^2$$

it follows

$$\begin{aligned} \max\{|x_1^{(k+1)} - x^*|, |x_2^{(k+1)} - x^*|\} &\leq q \max\{|x_1^{(k)} - x^*|^2, |x_2^{(k)} - x^*|^2\} \\ &= q(\max\{|x_1^{(k)} - x^*|, |x_2^{(k)} - x^*|\})^2, \end{aligned}$$

that is

$$|X^{(k+1)} - x^*| \leq q|X^{(k)} - x^*|^2,$$

which proves the theorem in this case. Since the case $X^{(0)} \subset D''$ leads after the first iteration to one of the previous cases, the proof of Theorem 4 is complete.

Following Alefeld and Herzberger [1, p. 97] we may modify the iteration scheme (10) as follows:

choose $X^{(0)} \subset D$, $X^{(0)} \ni x^*$, and

let $X^{(k+1)} = Y^{(k)} - F(Y^{(k)})/F'(Y^{(k)})$,

where

$$Y^{(k)} = \begin{cases} [x_1^{(k)}, m(X^{(k)})], & \text{if } f(x_1^{(k)})f(m(X^{(k)})) < 0 \\ m(X^{(k)}, x_2^{(k)}], & \text{if } f(x_1^{(k)})f(m(X^{(k)})) > 0 \\ X^{(k)}, & \text{otherwise} \end{cases} \quad (16)$$

and $m(X)$ denotes the midpoint of X .

In the next section we give a numerical comparison between the interval iteration scheme (10) and Moore's method [5, Sect. 7.2]:

$$\begin{aligned} & \text{choose } X^{(0)} \ni x^* \\ & \text{let } X^{(k+1)} = (m(X^{(k)}) - F(m(X^{(k)})) / F'(X^{(k)})) \cap X^{(k)} \end{aligned} \quad (17)$$

We also give a comparison between the modified interval iteration method (16) and the modified (after Alefeld and Herzberger) Moore's method:

$$\begin{aligned} & \text{choose } X^{(0)} \ni x^*, \\ & \text{let } X^{(k+1)} = \left(m(X^{(k)}) - \frac{f(m(X^{(k)}))}{F'(Y^{(k)})} \right) \cap X^{(k)}, \\ & \text{where} \\ & Y^{(k)} = \begin{cases} [x_1^{(k)}, m(X^{(k)})], & \text{if } f(x_1^{(k)})f(m(X^{(k)})) < 0 \\ [m(X^{(k)}), x_2^{(k)}], & \text{if } f(x_1^{(k)})f(m(X^{(k)})) > 0 \\ X^{(k)}, & \text{otherwise.} \end{cases} \end{aligned} \quad (18)$$

The numerical calculations in the next section are performed by Ivan Nedkov. Thereby a precompiler for extended interval arithmetic created by Nikolai Dushkov has been used.

7. Numerical Results

For the comparison of the methods we use two examples of equations from [1, p. 97–99]. We shall first compare numerically the methods (10) and (17).

Example 1. For the solution of the equation

$$f(x) = x^2(x^2/3 + 2 \sin x) - 3/19 = 0$$

with $X^{(0)} = [0.1, 1]$ we obtain the following interval approximations:

a) by means of Moore's method (17):

$$\begin{aligned} X^{(1)} &= [0.000000000000, 0.513607416303] \\ X^{(2)} &= [0.346173111779, 0.513607416303] \\ X^{(3)} &= [0.376596765306, 0.405984406594] \\ X^{(4)} &= [0.392303634719, 0.392471163891] \\ X^{(5)} &= [0.392379503865, 0.392379510728] \\ X^{(6)} &= [0.392379507136, 0.392379507136] \end{aligned}$$

b) by means of the interval Newton method (10):

$$\begin{aligned} X^{(1)} &= [0.120037044750, 0.680133943621] \\ X^{(2)} &= [0.161502933620, 0.496973170075] \\ X^{(3)} &= [0.236010152023, 0.413088339734] \end{aligned}$$

$$X^{(4)} = [0.327467643386, 0.393417674107]$$

$$X^{(5)} = [0.381767247666, 0.392382294247]$$

$$X^{(6)} = [0.392089764461, 0.392379507156]$$

$$X^{(7)} = [0.392379289318, 0.392379507136]$$

$$X^{(8)} = [0.392379507136, 0.392379507136]$$

This shows that an accuracy of 10^{-12} is achieved with 6 iterations by means of (17) and with 8 iterations by means of (10).

Example 2. For the solution of the equation

$$p(x) = x(x^9 - 1) - 1$$

with $X^{(0)} = [1, 1.5]$ we obtain the following approximations:

a) by means of (17):

$$X^{(1)} = [0.100000000000, 0.123157901169]$$

$$X^{(2)} = [0.101853906531, 0.110215348995]$$

$$X^{(3)} = [0.107180976833, 0.108476244466]$$

$$X^{(4)} = [0.107564709432, 0.107593118087]$$

$$X^{(5)} = [0.107576603950, 0.107576609732]$$

$$X^{(6)} = [0.107576606608, 0.107576606608]$$

b) by means of (10):

$$X^{(1)} = [0.100260801352, 0.135612883879]$$

$$X^{(2)} = [0.100894156840, 0.123492229604]$$

$$X^{(3)} = [0.102286076683, 0.114352015277]$$

$$X^{(4)} = [0.104657759845, 0.109173023027]$$

$$X^{(5)} = [0.106892552883, 0.107682466778]$$

$$X^{(6)} = [0.107550142704, 0.107577098984]$$

$$X^{(7)} = [0.107576574583, 0.107576606619]$$

$$X^{(8)} = [0.107576606608, 0.107576606608]$$

showing again that Moore's method uses less iterations for a calculation of the zero with an error $\leq 10^{-12}$.

We shall next compare the methods (16) and (18) using the same equations as above.

Example 3. For the equation

$$f(x) = x^2(x^2/3 + 2 \sin x) - 3/19 = 0$$

with $X^{(0)} = [0.1, 1]$ we obtain:

a) by means of (18):

$$X^{(1)} = [0.100000000000, 0.433580540100]$$

$$X^{(2)} = [0.339533432220, 0.433580540100]$$

$$X^{(3)} = [0.391237711372, 0.392469297358]$$

$$X^{(4)} = [0.392378544687, 0.392380226495]$$

$$X^{(5)} = [0.3923795071359, 0.3923795071364]$$

b) by means of (16):

$$X^{(1)} = [0.164098277476, 0.433580540100]$$

$$X^{(2)} = [0.358156977623, 0.396230123115]$$

$$X^{(3)} = [0.391501480063, 0.392417480786]$$

$$X^{(4)} = [0.392378966757, 0.392379510878]$$

$$X^{(5)} = [0.392379507136, 0.392379507136]$$

Example 4. For the solution of the equation

$$p(x) = x(x^9 - 1) - 1$$

with $X^{(0)} = [1, 1.5]$ we obtain the following approximations:

a) by means of (18):

$$X^{(1)} = [0.100000000000, 0.115390928100]$$

$$X^{(2)} = [0.107452573315, 0.107577227002]$$

$$X^{(3)} = [0.107576435513, 0.107576774994]$$

$$X^{(4)} = [0.107576606608, 0.107576606608]$$

b) by means of (16):

$$X^{(1)} = [0.101360436753, 0.115390928100]$$

$$X^{(2)} = [0.105788147041, 0.107603929621]$$

$$X^{(3)} = [0.107541095006, 0.107576639509]$$

$$X^{(4)} = [0.107576592680, 0.107576606608]$$

$$X^{(5)} = [0.107576606608, 0.107576606608]$$

References

- [1] Alefeld, G., Herzberger, J.: Einführung in die Intervallrechnung. Mannheim: Bibliographisches Institut 1974.
- [2] Caprani, O., Madsen, K.: Contraction mappings in interval analysis. BIT **15**, 362 – 366 (1975).
- [3] Markov, S.: Extended interval arithmetic. Compt. Rend. Acad. Bulg. Sci. **31**, 163 – 166 (1978).
- [4] Markov, S.: Extended interval arithmetic and some applications. Freiburger Intervall-Berichte 78,4, Institut für Angewandte Mathematik, Univ. Freiburg i. Br. 1978.

- [5] Moore, R. E.: Interval Analysis. Englewood Cliffs, N. J.: Prentice-Hall 1966.
- [6] Nickel, K.: Intervall-Mathematik. ZAMM **58**, T72 – T85 (1978).

S. M. Markov
Mathematisches Institut
Bulgarische Akademie der Wissenschaften
Postfach 373
Sofia, Bulgaria