On the Algebraic Properties of Convex Bodies

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Abstract

The algebraic properties of the convex bodies are studied. A theorem of H. Rådström for embedding of convex bodies in a normed vector space is generalized.

Key words: convex bodies, Minkowski operations, isomorphic embedding, quasilinear system.

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1 Introduction

Convex bodies form an abelian cancellative semigroup with respect to addition. By means of the well-known extension method (used e. g. when defining negative numbers) abelian cancellative semigroups can be isomorphically extended (embedded) into groups. Clearly, it is more convenient to compute within a group than whithin a semigroup, where the elements are not invertible in general. Convex bodies and intervals have been increasingly used in set-valued (convex), resp. interval analysis. The application of the extension method to systems of convex bodies and intervals has been studied by a number of authors. Our study is closely related to the work of H. Rådström [16], where the additive semigroup of convex bodies is embedded in a group and multiplication by real scalar is extended in a suitable way so that the group becomes a vector space [16]. Here we proceed similarly; using the classical extension method and a natural extension of the multiplication by real scalar we obtain a more general, so-called q-linear space, which contains the linear space constructed by H. Rådström. A simple distributivity relation is found in the q-linear space.

The paper is structured as follows: Section 2 is devoted to the operation addition of convex bodies and its algebraic properties. In Section 3 we consider multiplication by real scalar and its algebraic properties especially in relation to addition. In Section 4 some properties of the Hausdorff metric and the inclusion relation are discussed. Section 5 is devoted to a theorem of H. Rådström. The main result — an analogue of the theorem of H. Rådström producing isomorphic embedding — is given in Section 6.

In the paper we also present some well-known facts about the algebraic operations for convex bodies, which may be useful for a better understanding of the obtained results, although having no direct relation to these results.

2 Addition and Summability

By $\mathbb{E} = \mathbb{E}^n$, $n \geq 1$, we denote the n-dimensional real Euclidean vector space with origin 0. The field of reals is denoted by $\mathbb{R}$. A convex compact subset of $\mathbb{E}$ is called *convex body (in* $\mathbb{E}$) or just *body*; a convex body need not have necessarily interior points, e. g. a line segment and a single point in $\mathbb{E}$ are convex bodies [18]. The set of all convex bodies (in $\mathbb{E}$) will be denoted by $\mathcal{K} = \mathcal{K}(\mathbb{E})$. The set of all single points in $\mathcal{K}$ will be also denoted by $\mathbb{E}$, the elements of $\mathbb{E}$ are sometimes called degenerate elements (of $\mathcal{K}$). In the case $n = 1$ the elements of $\mathcal{K}(\mathbb{E})$ are compact intervals on the real line; we shall call them simply *intervals*.

Addition. The sum of two convex bodies $A, B \in \mathcal{K}$ (sometimes called Minkowski sum [5]) is defined by

$$A + B \quad = \quad \{c \mid c = a + b,\ a \in A,\ b \in B\}, \quad A, B \in \mathcal{K}. \tag{1}$$

Clearly, (1) defines an algebraic operation in $\mathcal{K}$ called *addition* and $\mathcal{K}$ is closed under this operation. For all $A, B, C \in \mathcal{K}$ we have

$$(A + B) + C \quad = \quad A + (B + C), \tag{2}$$

$$A + B \quad = \quad B + A, \tag{3}$$

$$A + 0 \quad = \quad A, \tag{4}$$

$$A + C = B + C \quad \Longrightarrow \quad A = B. \tag{5}$$

The verification of (2)–(4) is straightforward, for a proof of relation (5) see [16].

If $B = \{b\} \in \mathbb{E}$ is degenerate and $A \in \mathcal{K}$, then we may write $A + B$ as $A + b$, and $A + \{-b\}$ is written as $A + (-b)$ or as $A - b$. Of course, the element $-b$ is the opposite to b in $\mathbb{E}$, that is $b + (-b) = 0$. A body of the form $A = x + B$, for $x \in \mathbb{E}$, $B \in \mathcal{K}$, is called a *translate* of B (*by the vector* x). Using translates, we can write (1) in the form

$$A + B = \bigcup_{b \in B} (A + b). \tag{6}$$

Equation (6) gives a geometrical insight for the sum, especially in relation to the Minkowski difference to be defined next.

Minkowski difference. Let A, $B \in \mathcal{K}$. The expression

$$A \stackrel{*}{-} B = \bigcap_{b \in B} (A - b), \tag{7}$$

is defined whenever the right-hand side is not empty; it has been introduced for convex bodies and studied by H. Hadwiger under the name *Minkowski difference* [4], [5].

Expression (7) defines a partial operation in the set $\mathcal{K}$. If the intersection in the right-hand side of (7) is empty, then $A \stackrel{*}{-} B$ is not defined, we shall denote this case by $A \stackrel{*}{-} B = \emptyset$; alternatively the symbol $A \stackrel{*}{-} B \neq \emptyset$ means that the convex body $A \stackrel{*}{-} B$ is well defined by (7).

Note that the right-hand side of (7) is not empty, if and only if there exists a vector $t \in \mathbb{E}$, such that $t + B \subset A$. We shall denote the induced partial order relation between A and B by:

$$B \leq_M A \Longleftrightarrow A \stackrel{*}{-} B \neq \emptyset \Longleftrightarrow \exists t : t + B \subset A.$$

In particular, $B \subset A$ implies $B \leq_M A$. The following equivalent presentation of (7) holds

$$A \stackrel{*}{-} B \quad = \quad \{x \in \mathbb{E} \mid x + B \subset A\}, \tag{8}$$

saying that $A \stackrel{*}{-} B$ is the set of all vectors $x \in \mathbb{E}$, such that the translate of B by x belongs to A. Note the important equality $A \stackrel{*}{-} A = 0$ for all $A \in \mathcal{K}$. From (8) we have for $A,\ B \in \mathcal{K}$ [5]

$$(A \stackrel{*}{-} B) + B \subset A. \tag{9}$$

Also, we have [5]: $(A \stackrel{*}{-} B) \stackrel{*}{-} C = A \stackrel{*}{-} (B + C)$; other useful equality is: $(A + B) \stackrel{*}{-} B = A$.

Summability. For two nonempty convex bodies $A, B \in \mathcal{K}$ we say that B is a *summand of* A if there exists $X \in \mathcal{K}$, such that $A = B + X$; of course, X is also a summand of A. We see from (8) and (9) that, if B is a summand of A, then $A \stackrel{*}{-} B$ is also a summand of A (see Lemma 3.1.8, [18]). In other words, if B is a summand of A, then $X = A \stackrel{*}{-} B$ is a solution of $X + B = A$, and in this case inclusion (9) becomes an equality:

$$(A \stackrel{*}{-} B) + B = A \Longleftrightarrow B \text{ is a summand of } A. \tag{10}$$

For $A,\ B \in \mathcal{K}$ we say that B slides freely inside A, if to each boundary point $a \in A$ there exists $x \in \mathbb{E}$, such that the translate of B by x contains a and belongs to A: $a \in B + x \subset A$. The following proposition gives a geometrical insight of the relation "B is a summand of A":

Proposition 1 (Th. 3.2.2 of [18]) *Let $A,\ B \in \mathcal{K}$. Then B is a summand of A, if and only if B slides freely inside A.*

Proof. If $A = B + X$ and $a \in A$, there exists $b \in B$ and $x \in X$ such that $a = b + x$, hence $a \in B + x \subset A$. Vice versa, let B slides freely inside A. Let $a \in A$ be a boundary point of A. By assumption, there exists $x \in \mathbb{E}$, such that $a \in B + x \subset A$. Then $x \in A \stackrel{*}{-} B$ and $a \in B + (A \stackrel{*}{-} B)$. Hence, the boundary of A belongs to $(A \stackrel{*}{-} B) + B$ and therefore $A = (A \stackrel{*}{-} B) + B$. By (10) we obtain that B is a summand of A. $\square$

Example. Let A, B be the unit square, resp. the unit ball in $\mathbb{E}^3$. We have $B \subset A$ and (9): $(A \stackrel{*}{-} B) + B \subset A$. The latter inclusion is strong in the sense that B is not a summand of A (B does not slide freely in A). The situation does not change if instead of B we take a ball with smaller radius, unless B is not degenerate.

The equation $A + X = B$ may have a solution for certain pairs A, B. The solution X of $B + X = A$, if existing, is unique. Indeed, by definition, there is a $X \in \mathcal{K}$, such that $A = B + X$. Assume that $X' \in \mathcal{K}$, with $X' \neq X$ is such that $A = B + X'$. Then we have $B + X = B + X'$, which by the cancellation law (5) implies $X = X'$, a contradiction.

It follows from (10) that the solution of $A = B + X$, if existing, is exactly $X = A \stackrel{*}{-} B$.

Obviously, any summand of a degenerate (one-point) body is a degenerate body itself. Hence, from $X + Y = 0$ it follows that X and Y are degenerate bodies, and $Y = -X$. In $\mathcal{K}$ the equation $A + X = 0$ is not solvable if $A \in \mathcal{K} \setminus \mathbb{E}$.

In what follows we shall symbolically denote the partial order relation "B is a summand of A" by $B \leq_\Sigma A$, or $A \geq_\Sigma B$. "B is not a summand of A" will be denoted by $B \not\leq_\Sigma A$. From Proposition 1, we see that $B \leq_\Sigma A$ implies $B \leq_M A$; however, the inverse is not true. In Example 1 we have $B \leq_M A$, but not $B \leq_\Sigma A$.

If $A, B \in \mathcal{K}$, then there exist the following possibilities: i) $B \leq_\Sigma A$ and $A \not\leq_\Sigma B$, denoted $B <_\Sigma A$; ii) $A \leq_\Sigma B$ and $B \not\leq_\Sigma A$, denoted $A <_\Sigma B$; iii) $A \leq_\Sigma B$ and $B \leq_\Sigma A$, denoted $A =_\Sigma B$; iv) $A \not\leq_\Sigma B$ and $B \not\leq_\Sigma A$.

If one of the cases i)–iii) holds, we say that the pair (A, B) is Σ-*comparable*. The set of all Σ-comparable pairs is denoted by $\mathcal{L}_\Sigma$. Clearly, if $(A, B) \in \mathcal{L}_\Sigma$, then at least one of the two convex bodies $A \stackrel{*}{-} B$, $B \stackrel{*}{-} A$ is well defined.

In case iii) there exists $X \in \mathcal{K}$, such that $A = B + X$, and there exists $Y \in \mathcal{K}$, such that $B = A + Y$. Summing up both equations we obtain $A + B = (B + X) + (A + Y) = (A + B) + X + Y$, and by (5), $X + Y = 0$, hence $X, Y \in \mathbb{E}$, $Y = -X$. Thus, in case iii) A is a translate of B by some X, and B is a translate of A by $-X$.

Note that, if $A, B \in \mathcal{K}(\mathbb{E})$, $(A, B) \in \mathcal{L}_\Sigma$, then for the equations

$$B + X = A, \tag{11}$$
$$A + Y = B, \tag{12}$$

exactly one of the possibilities i)–iii) mentioned above holds true, i. e.:

1) Case $B <_\Sigma A$: there exists a unique nondegenerate convex body $X \in \mathcal{K} \setminus \mathbb{E}$ satisfying (11); equation (12) is not solvable.

2) Case $A <_\Sigma B$: there exists a unique nondegenerate convex body $Y \in \mathcal{K} \setminus \mathbb{E}$ satisfying (12); equation (11) is not solvable.

3) Case $A =_\Sigma B$: both (11), (12) are solvable for X, resp. Y, and we have $Y = -X \in \mathbb{E}$.

In $\mathbb{E}^1$ the convex bodies are intervals on the real line. In $\mathbb{E}^1$ all elements are Σ-comparable, $(A, B) \in \mathcal{L}_\Sigma$, hence exactly one of the cases 1)–3) is satisfied [9]. If we drop out the condition $(A, B) \in \mathcal{L}_\Sigma$ in the general case, then we can only state that:

Proposition 2 *For every two $A, B \in \mathcal{K}$ each of the equations (11), (12) may have at most one solution.*

3 The Quasidistributive Law

Multiplication by real scalar is defined by

$$\alpha * B \quad = \quad \{c \mid c = \alpha b,\ b \in B\}, \quad B \in \mathcal{K},\ \alpha \in \mathbb{R}. \tag{13}$$

Recall some properties of (13). For $A, B, C \in \mathcal{K}$, $\alpha, \beta, \gamma \in \mathbb{R}$, we have:

$$\begin{aligned}
\gamma * (A + B) \quad &= \quad \gamma * A + \gamma * B, && (14)\\
\alpha * (\beta * C) \quad &= \quad (\alpha\beta) * C, && (15)\\
1 * A \quad &= \quad A, && (16)\\
(\alpha + \beta) * C \quad &= \quad \alpha * C + \beta * C, \quad \alpha\beta \geq 0. && (17)
\end{aligned}$$

Relations (14)–(16) are easily verified. To verify (17) recall that a subset $C \subset \mathbb{E}$ is convex if for $x, y \in C$:

$$\alpha x + \beta y \in C, \qquad \alpha, \beta \geq 0,\ \alpha + \beta = 1.$$

If C is convex and $x \in \alpha * C + \beta * C$, then $x = \alpha a + \beta b$ with some $a, b \in C$, hence

$$x = (\alpha + \beta)\left(\frac{\alpha}{\alpha + \beta} a + \frac{\beta}{\alpha + \beta} b\right) \in (\alpha + \beta) * C.$$

Therefore $\alpha * C + \beta * C = (\alpha + \beta) * C$, $\alpha, \beta \geq 0$ (note that the inclusion $\alpha * C + \beta * C \supset (\alpha + \beta) * C$ is trivially fulfilled even for nonconvex C).

Relation (17) is not valid for $\alpha\beta < 0$ and C nondegenerate. However, we can easily express $(\alpha + \beta) * C$ in terms of $\alpha * C$ and $\beta * C$ for $\alpha\beta < 0$. Indeed, using (10), from (17) we have for $\alpha > 0$, $\beta > 0$: $\alpha * C = (\alpha + \beta) * C \stackrel{*}{-} \beta * C$. Substituting $\alpha + \beta = \lambda > 0$ (and hence $\lambda > \beta > 0$) we obtain:

$$(\lambda - \beta) * C = \lambda * C \stackrel{*}{-} \beta * C, \quad \lambda > \beta > 0. \tag{18}$$

Substituting in (18) $\beta = -\mu$, $\mu < 0$, we have $(\lambda + \mu) * C = \lambda * C \stackrel{*}{-} (-\mu) * C$, $\lambda > -\mu > 0$ (cf. also [5]), [13, 14]). Using the original notation α, β we can write:

$$(\alpha + \beta) * C = \alpha * C \stackrel{*}{-} (-\beta) * C, \quad \alpha > -\beta > 0, \tag{19}$$

which can be written more symmetrically as

$$(\alpha + \beta) * C = \begin{cases} \alpha * C \stackrel{*}{-} (-\beta) * C, & \text{if } \alpha\beta < 0, |\alpha| \geq |\beta|, \\ \beta * C \stackrel{*}{-} (-\alpha) * C, & \text{if } \alpha\beta < 0, |\alpha| < |\beta|. \end{cases}$$

Combining relation (17) and the above formula we can write a general expression of $(\alpha + \beta) * C$ in terms of $\alpha * C$ and $\beta * C$, which is valid for all $\alpha, \beta \in \mathbb{R}$:

$$(\alpha + \beta) * C = \begin{cases} \alpha * C + \beta * C, & \text{if } \alpha\beta \geq 0, \\ \alpha * C \stackrel{*}{-} (-\beta) * C, & \text{if } \alpha\beta < 0, |\alpha| \geq |\beta|, \\ \beta * C \stackrel{*}{-} (-\alpha) * C, & \text{if } \alpha\beta < 0, |\alpha| < |\beta|. \end{cases} \tag{20}$$

Relation (17) or its corollary (20) will be further refered to as *quasidistributive law*.

The following equality is also valid [5]:

$$\gamma * (A \stackrel{*}{-} B) = \gamma * A \stackrel{*}{-} \gamma * B,$$

showing that an analogue of the first distributive law (14) holds for the Minkowski difference.

Negation. The operator neg: $\mathcal{K} \longrightarrow \mathcal{K}$ defined by $\operatorname{neg}(A) = (-1) * A = \{-a \mid a \in A\}$, $A \in \mathcal{K}$, is called *negation*, and will be symbolically denoted by $\neg A$.

For brevity, we denote for $A,\ B \in \mathcal{K}$

$$A \neg B \equiv A + (\neg B) = \{a - b \mid a \in A,\ b \in B\}; \tag{21}$$

the operation $A \neg B$ is called *(outer) subtraction*.

We have $\neg(\gamma * A) = (-1) * (\gamma * A) = (-\gamma) * A = \gamma * (\neg A)$ for any real γ and $A \in \mathcal{K}$.

Remarks. Instead of "$\neg$" the symbol "$-$" is widely used in the literature on convex, set-valued and interval analysis (see e. g. [1], [7], [17], [18], etc.). It should be kept in mind that $A \neg A \neq 0$ for $A \in \mathcal{K} \setminus \mathbb{E}$. Since the notation "$-$" is usually associated with the equality $A - A = 0$, for all A, to avoid confusion in this work we write "$\neg$" instead of "$-$". Using the symbol "$\neg$" we shall also avoid confusion with the opposite in the extended space of pairs of convex bodies to be introduced in Section 5. In mathematical morphology the outer subtraction (21) is called *dilatation*, whereas the Minkowski subtraction is called *erosion* [12], [15].

Symmetric bodies. An element $A \in \mathcal{K}$ is called *symmetric with respect to the origin*, if $x \in \mathbb{E}$, $x \in A$, implies $-x \in A$.

The set of all symmetric convex bodies is denoted by $\mathcal{K}_S$. We have $\mathcal{K}_S = \{A \in \mathcal{K} \mid A = \neg A\}$, i. e. $A \in \mathcal{K}$ is symmetric, if and only if $A = \neg A$. For $A \in \mathcal{K}$, the set $A \neg A$ is called the difference body of A (see [18], p. 127). For $A \in \mathcal{K}$, we have $A \neg A \in \mathcal{K}_S$. Indeed, we have $\neg(A \neg A) = \neg A + A = A \neg A$.

Proposition 3 *The following two conditions for symmetricity of $A \in \mathcal{K}$ are equivalent:*

i) $A = \neg A$;

ii) there exists $Z \in \mathcal{K}$, such that $A = Z \neg Z$.

Proof. i) Let $A = \neg A$. Assume $t \in \mathbb{E}$ and set $Z = A/2 + t$, where $A/2 = (1/2) * A$. Using $A = \neg A$ we obtain $\neg Z = \neg A/2 - t = A/2 - t$. Hence $Z \neg Z = Z + (\neg Z) = (A/2 + t) + (A/2 - t) = A$. ii) Assume that $A = Z \neg Z$ for some $Z \in \mathcal{K}$. Then we have $\neg A = \neg(Z \neg Z) = \neg Z + Z = Z \neg Z = A$. $\square$

The element $A \in \mathcal{K}$ is called *t-symmetric*, with *center* $t \in \mathbb{E}$, if $(A - t) \in \mathcal{K}_S$. In other words, a t-symmetric element is a t-translate of a symmetric element. The latter can be considered as a special case of t-symmetric element, i. e. a 0-symmetric element.

Proposition 4 *Every t-symmetric convex body A is a translate of its negation $\neg A$.*

Proof. Let $A \in \mathcal{K}$ be t-symmetric. This means that $A - t$ is symmetric, that is $A - t = \neg(A - t) = \neg A + t$. This implies $\neg A + 2t = A$, hence A is a $(2t)$-translate of $\neg A$. $\square$

Let $A \in \mathcal{K}$ be t-symmetric, i. e. $A - t \in \mathcal{K}_S$. By Proposition 3 there exists $Z \in \mathcal{K}$ such that $A - t = Z \neg Z$. To find an explicit expression for Z, fix $s \in \mathbb{E}$ and set $Z = (A - t)/2 + s$; we obtain $Z = A/2 + s'$, $s' \in \mathbb{E}$. Thus $A - t = Z \neg Z = A/2 \neg A/2 = (A \neg A)/2$. We have $A - t = (A \neg A)/2$, that is, for any t-symmetric element $A \in \mathcal{K}$, its symmetric translate is $(A \neg A)/2$ (by the vector $-t$).

We shall end this section by proving directly that the terms appearing in the right-hand side of (20) are Σ-comparable. Due to this fact the expression in the left-hand side of (20) can be splitted into two terms for any choice of α and β. We first prove the following:

Lemma. Let $C \in \mathcal{K}$, $\alpha \in \mathbb{R}$, $\alpha \geq 0$. Then

$$\begin{aligned} C &\geq_\Sigma \alpha * C, \quad \text{if} \quad 0 \leq \alpha \leq 1, \\ C &\leq_\Sigma \alpha * C, \quad \text{if} \quad \alpha \geq 1. \end{aligned}$$

Proof. Let $0 \leq \alpha \leq 1$. We have to verify that $\alpha * C$ is a summand of C, that is $\alpha * C + X = C$ for some $X \in \mathcal{K}$. Take $X = (1 - \alpha) * C$. Substituting $\beta = 1 - \alpha \geq 0$ in (17) we obtain $\alpha * C + (1 - \alpha) * C = (\alpha + 1 - \alpha) * C = C$, showing that $C \geq_\Sigma \alpha * C$, for $\alpha \in [0, 1]$. Let $\alpha \geq 1$. We look for Y, such that $\alpha * C = Y + C$. Taking $Y = (\alpha - 1) * C$ we see that $C \leq_\Sigma \alpha * C$, for $\alpha \geq 1$. $\square$

The above lemma shows that for $\alpha \in (0, 1)$ the equation $C = \alpha * C + X$ has a solution $X = (1 - \alpha) * C$; for $\alpha \geq 1$ the equation $C + X = \alpha * C$ has a solution $X = (\alpha - 1) * C$.

Proposition 5 *Let α, $\beta \in \mathbb{R}$, $C \in \mathcal{K}$. If $\alpha\beta > 0$, then $(\alpha * C, \beta * C) \in \mathcal{L}_\Sigma$. If $\alpha\beta < 0$, then $(\neg(\alpha * C),\ \beta * C) \in \mathcal{L}_\Sigma$.*

Proof. Let $\alpha\beta > 0$, say $\alpha \geq \beta > 0$. We shall show that the pair $(\alpha * C, \beta * C)$ is Σ-comparable, i. e. either $\alpha * C \leq_\Sigma \beta * C$, or $\beta * C \leq_\Sigma \alpha * C$. By the above Lemma we have that C and $(\alpha/\beta) * C$, $\alpha/\beta > 1$ are Σ-comparable $(\alpha/\beta) * C \geq_\Sigma C$, that is $C + X = (\alpha/\beta) * C$ is solvable. Then $\beta * C + Y = \alpha * C$ is solvable, i. e. $\alpha * C \geq_\Sigma \beta * C$.

Let now $\alpha\beta < 0$. Without loss of generality we may assume that $\alpha \geq -\beta > 0$. By the Lemma we have that the pair (C, D) with $D = (-\alpha/\beta) * C = (\alpha/\beta) * (\neg C)$ is Σ-comparable, and $(\alpha/\beta) * (\neg C) \geq_\Sigma C$. This implies that the pair $(\neg(\alpha * C), \beta * C)$, resp. the pair $(\alpha * C, \neg(\beta * C))$ is Σ-comparable. $\square$

Note that $A \leq_\Sigma B$ does imply $\neg A \leq_\Sigma \neg B$, but does not necessarily imply $\neg A \leq_\Sigma B$. Due to this fact for some A, $B \in \mathcal{K}$ it may happen that $A \stackrel{*}{-} (\neg B) = \emptyset$ and $A \stackrel{*}{-} B \neq \emptyset$, or vice versa, both relations $A \stackrel{*}{-} B = \emptyset$ and $A \stackrel{*}{-} (\neg B) \neq \emptyset$ may hold true simultaneously.

Proposition 6 *Let* $(A, \neg B) \in \mathcal{L}_\Sigma$. *Then* $A \stackrel{*}{-} (\neg B) \leq_\Sigma A + B$.

Proof. Obviously $A \stackrel{*}{-} (\neg(B)$ is a summand of $A+B$. Indeed, if $\neg B \leq_\Sigma A$ we have $\neg B + (A \stackrel{*}{-} (\neg B)) = A$, hence $B \neg B + (A \stackrel{*}{-} (\neg B)) = A + B$. If $\neg A \leq_\Sigma B$, then $\neg A + (A \stackrel{*}{-} (\neg B)) = B$, and hence $A \neg A + (A \stackrel{*}{-} (\neg B)) = A + B$. $\square$

4 Some Properties of the Metric and the Inclusion Relation

A natural metric in $\mathcal{K}$ is the Hausdorff distance, defined for $X, Y \in \mathcal{K}$ by

$$d(X,Y) = \max\{\max_{x \in X} \min_{y \in Y} |x-y|, d(X,Y) = \max\{\max_{x \in Y} \min_{y \in X} |x-y|\},$$

or, equivalently,

$$d(X,Y) = \min_{\lambda \geq 0}\{X \subset Y + \lambda * B, Y \subset X + \lambda * B\},$$

where $|x-y|$, resp. B, is the distance, resp. the unit ball in $\mathbb{E}$. For a proof of the equivalency see [18], section 1.8.

The Hausdorff distance satisfies the following properties [16]:

$$d(A+C, B+C) = d(A,B), \tag{22}$$
$$d(\lambda * A, \lambda * B) = \lambda d(A,B),\ \lambda \geq 0. \tag{23}$$

Inclusion "$\subset$" is a partial order relation in $\mathcal{K}$. Inclusion is consistent with the arithmetic operations in $\mathcal{K}$ in the following sense: for $A, B, C, D \in \mathcal{K}$, $\alpha, \beta \in \mathbb{R}$:

$$A \subset C \iff A + B \subset C + B, \tag{24}$$
$$A \subset C,\ B \subset D \implies A + B \subset C + D, \tag{25}$$
$$A \subset C,\ B \supset D \implies A \stackrel{*}{-} B \subset C \stackrel{*}{-} D. \tag{26}$$

Relations (24), (25) are known as *isotonicity* of addition. Relation (26) is proved in [5]. Scalar multiplication is isotone in the sense that:

$$A \subset B \iff \lambda * A \subset \lambda * B, \lambda \in \mathbb{R}.$$

In particular, we have $A \subset B \iff \neg A \subset \neg B$. The following relation is also formulated in [5]:

$$(A+B) \stackrel{*}{-} (C+D) \supset (A \stackrel{*}{-} C) + (B \stackrel{*}{-} D). \tag{27}$$

A special case of (27) is:

$$(A+B) \stackrel{*}{-} C \supset A + (B \stackrel{*}{-} C). \tag{28}$$

Denote $A \bigvee B = \mathrm{conv}(A \bigcup B)$ (for the definition of convex hull $\mathrm{conv} : \mathcal{K} \longrightarrow \mathcal{K}$ see, e. g. [17] or [18]). We have [5]:

$$\begin{aligned}
(A \bigvee B) + C &= (A + C) \bigvee (B + C), \\
(A \bigcap B) + C &= (A + C) \bigcap (B + C), \\
(A \bigvee B) \stackrel{*}{-} C &= (A \stackrel{*}{-} C) \bigvee (B \stackrel{*}{-} C), \quad A \bigcap B = \emptyset, \\
(A \bigcap B) \stackrel{*}{-} C &= (A \stackrel{*}{-} C) \bigcap (B \stackrel{*}{-} C), \\
(A + B) \stackrel{*}{-} (B + C) &\supset (A \stackrel{*}{-} C) + (B \stackrel{*}{-} C).
\end{aligned}$$

Proposition 7 *Let for $A, B, C \in \mathcal{K}$, $A + B = C$ and $0 \in A$. Then $B \subset C$.*

Proof. Equation $A + B = C$, that is $\bigcup_{a \in A} a + B = C$, means $a + B \subset C$ for all $a \in A$. Hence for $a = 0$, $B = 0 + B \subset C$. $\square$

The next proposition is closely related to Proposition 6.

Proposition 8 *Let $(A, \neg B) \in \mathcal{L}_\Sigma$. Then $A \stackrel{*}{-} (\neg B) \subset A + B$.*

Proof. From Proposition 6 we know that $A \stackrel{*}{-} (\neg B)$ is a summand of $A + B$. If $\neg B \leq_\Sigma A$ we have $\neg B + (A \stackrel{*}{-} (\neg B)) = A$, hence $B \neg B + (A \stackrel{*}{-} (\neg B)) = A + B$. If $\neg A \leq_\Sigma B$, then $\neg A + (A \stackrel{*}{-} (\neg B)) = B$, and hence $A \neg A + (A \stackrel{*}{-} (\neg B)) = A + B$. Since $A \neg A \ni 0$, resp. $B \neg B \ni 0$, we have $A \stackrel{*}{-} (\neg B) \subset A + B$, using Proposition 7 and $A \stackrel{*}{-} (\neg B) \leq_\Sigma A + B$. $\square$

Propositions 6 and 8 can be generalized for all A, B, for which $A \stackrel{*}{-} (\neg B)$ is defined (not necessarily $(A, \neg B) \notin \mathcal{L}_\Sigma$, see [15]).

The system of convex bodies with addition, multiplication by real scalar, inclusion and metric will be denoted $(\mathcal{K}, \mathbb{E}, +, \mathbb{R}, *, \subset, \| \cdot \|)$. Computations in this system are hampered by the fact that nondegenerate convex bodies are not invertible. In the next section we show that this can be overcome by the extension method.

5 A Theorem of H. Rådström

Due to (2), (3), (5) the set $(\mathcal{K}, +)$ is an abelian cancellative (a. c.) semigroup with respect to addition "+". Moreover, the a. c. semigroup is an a. c. monoid $(\mathcal{K}, +, 0)$, that is, there exists a neutral element "0" in $\mathcal{K}$, such that (4) holds.

Using the extension method (see e. g. [3], [8], [16], we can embed isomorphically any a. c. monoid $(\mathcal{Q}, +, 0)$ into a group $(\mathcal{G}, +)$; we briefly recall the method below.

The extension method. Let $(\mathcal{Q}, +, 0)$ be an a. c. monoid. Define $\mathcal{G} = (\mathcal{Q} \times \mathcal{Q})/\rho$ to be the set of pairs[1] (A, B), $A,\ B \in \mathcal{Q}$, factorized by the equivalence relation $\rho : (A, B)\rho(C, D) \iff A + D = B + C$. Define addition in $\mathcal{G}$ by means of: $(A, B) + (C, D) = (A + C, B + D)$.

[1]instead of pairs (A, B) one often uses formal differences $A - B$

Denote the equivalence class in the group $\mathcal{G}$, represented by the pair (A, B), again by (A, B), hence we shall write $(A, B) = (A+X, B+X)$. The null element of $\mathcal{G}$ is the class (Z, Z); due to the existence of null element, we have $(Z, Z) = (0, 0)$. The opposite element to $(A, B) \in \mathcal{G}$ is denoted by $-(A, B)$. It is easy to see that $-(A, B) = (B, A)$; indeed $(A, B) + (-(A, B)) = (A, B) + (B, A) = (A + B, B + A) = (0, 0)$. Instead of $(A, B) + (-(C, D))$, we may write $(A, B) - (C, D)$; we have $(A, B) - (C, D) = (A, B) + (D, C) = (A + D, B + C)$. The system $(\mathcal{G}, +, 0, -)$ obtained by the extension method is an abelian group and is unique up to isomorphism.

To embed isomorphically $\mathcal{Q}$ into $\mathcal{G}$ we identify $A \in \mathcal{Q}$ with the equivalence class $(A, 0) = (A + X, X)$, $X \in \mathcal{Q}$. Thus all "proper elements" of $\mathcal{G}$ are pairs (U, V), $U,\ V \in \mathcal{Q}$, such that $V + Y = U$ for some $Y \in \mathcal{K}$, i. e. $(U, V) = (V + Y, V) = (Y, 0)$.

In an a. c. monoid $(\mathcal{Q}, +, 0)$ the set $\mathcal{Q}_0$ of all invertible elements form a group $(\mathcal{Q}_0, +, 0, -)$ which is a subgroup of the monoid (in the case of convex bodies $\mathcal{Q}_0 = \mathbb{E}$). Since this subgroup plays important role and includes the neutral element, we shall sometimes denote a monoid $(\mathcal{Q}, +, 0)$ also by $(\mathcal{Q}, \mathcal{Q}_0, +)$.

The next proposition shows when an element of $\mathcal{G}$ can be presented in one of the forms $(U, 0)$ or $(0, V)$.

Proposition 9 *Let* $(A, B) \in \mathcal{L}_\Sigma$. *Then*

$$(A, B) = \begin{cases} (A \stackrel{*}{-} B, 0), & \text{if } B \le_\Sigma A, \\ (0, \neg(A \stackrel{*}{-} B)), & \text{if } B >_\Sigma A. \end{cases}$$

Proof. By definition, if $B \le_\Sigma A$, then $B + (A \stackrel{*}{-} B) = A$. Hence, $(A, B) = (B + (A \stackrel{*}{-} B), B) = (A \stackrel{*}{-} B, 0)$. The case $B >_\Sigma A$ is treated analogously.

Assume that a multiplication by real scalar "$*$" is defined on the monoid $(\mathcal{Q}, \mathcal{Q}_0, +)$, satisfying (14)–(17). The algebraic system $(\mathcal{Q}, \mathcal{Q}_0, +, \mathbb{R}, *)$ is called a *quasilinear system*; for more details see [11].

Consider now the extension of "$*$" into $\mathcal{G}$. A natural definition of multiplication by real scalar in $\mathcal{G}$ is

$$\gamma * (A, B) = (\gamma * A, \gamma * B), \quad A,\ B \in \mathcal{Q}, \quad \gamma \in \mathbb{R}. \tag{29}$$

In particular, for $\gamma = -1$ we obtain negation in $\mathcal{G}$:

$$\neg(A, B) = (-1) * (A, B) = (\neg A, \neg B), \quad A,\ B \in \mathcal{Q}, \quad \gamma \in \mathbb{R}. \tag{30}$$

It is easy to see that $\mathcal{G}$ is not a vector space under "$*$" defined by (29). The possibility to obtain a vector space using other multiplication by scalar (distinct from (29)) has been studied in [16], where the following operation has been proposed:

$$\begin{aligned} \gamma \cdot (A, B) &= \begin{cases} (\gamma * A, \gamma * B), & \gamma \ge 0, \\ (|\gamma| * B, |\gamma| * A), & \gamma < 0, \end{cases} \\ &= \begin{cases} (\gamma * A, \gamma * B), & \gamma \ge 0, \\ ((-\gamma) * B, (-\gamma) * A), & \gamma < 0. \end{cases} \end{aligned} \tag{31}$$

Note that $(-1) \cdot (A, B) = (B, A)$ is the opposite in $\mathcal{G}$, i. e. for $y \in \mathcal{G}$ we have : $(-1) \cdot y = 0$; hence $y + (-1) \cdot y = 0$.

Below we formulate a result by H. Rådström (see [16], Theorem 1) in slightly modified form:

Proposition 10 *(Theorem of H. Rådström) Let $\mathcal{Q}$ be an a. c. monoid, i. e. properties (2)–(5) are satisfied. Then:*

A. Using the extension method $\mathcal{Q}$ can be embedded in a group $\mathcal{G}$, so that any element $A \in \mathcal{Q}$ is identical with $(A, 0) \in \mathcal{G}$. The group $\mathcal{G}$ can be chosen so as to be minimal in the following sense: if $\mathcal{G}'$ is any group in which $\mathcal{Q}$ is embedded, then $\mathcal{G}$ is isomorphic to a subgroup of $\mathcal{G}'$ containing $\mathcal{Q}$;

B. If a multiplication by real scalar "$*$" is defined on $\mathcal{Q}$, satisfying (14)–(17), and a multiplication by real scalar "$\cdot$" is defined [2] on $\mathcal{G}$ by means of (31), then $(\mathcal{G}, +, \mathbb{R}, \cdot)$ is a vector space and for any real $\lambda \geq 0$ and $A \in \mathcal{Q}$ the product $\lambda \cdot (A, 0)$ is identical with $\lambda * A$ on $\mathcal{Q}$.

C. If a metric $d(A, B)$ is given on $\mathcal{Q}$ satisfying: (22), (23) and if $A + B$ and $\lambda * A$ are continuous operations in the topology defined by d, then for $(A, B) \in \mathcal{G}$ the function $\delta((A, B), (C, D)) = d(A + D, B + C)$ defines a metric on $\mathcal{G}$, thereby $\delta((A, 0), (B, 0)) = d(A, B)$. The metric δ is homogeneous and invariant under translation, i. e. for $x, y \in \mathcal{G}$: i) $\delta(\lambda \cdot x, \lambda \cdot y) = \lambda \delta(x, y)$, $\lambda \geq 0$, ii) $\delta(x + z, y + z) = \delta(x, y)$. Also: $\delta(x, y) = \delta(x + (-1) \cdot y, 0)$, so that we can put $\delta(x, y) = \| x + (-1) \cdot y \|$; the function $\| z \| = \delta(z, 0)$ is a norm in $\mathcal{G}$, making $\mathcal{G}$ a normed linear space.

It is easy to see, that the system $(\mathcal{G}, +, \mathbb{R}, \cdot, \| \cdot \|)$ is a normed linear space.

We note that (31) is not an isomorphic extension of (13). Expression (31) is chosen is such a way that $(-1) \cdot a$, $a \in \mathcal{G}$, is the opposite in $\mathcal{G}$, which does not produce meaningfull results when multi[plying by negative numbers. In the next section we use expression (29) to extend the multiplication by scalar arriving thus to an isomorphic extension of the space of convex bodies.

6 An Isomorphic Extension of the Space of Convex Bodies

Our next proposition can be considered as generalization of Proposition 10. The basic idea is to take multiplication by real scalar in the group $\mathcal{G}$ according to (29) and to investigate the obtained algebraic system with respect to distributivity. As result, we obtain a simple second-distributivity-type relation. We also obtain that the multiplication (31) used by H. Rådström is involved in the obtained system (as composite operation); however, (29) is not involved in the vector space induced by (31). We shall also incorporate an extension of the important inclusion relation in the spirit of [6], where this has been done for intervals.

To facilitate the formulation of our main result we shall introduce a brief symbolic notation for the composition of the operators opposite and negation.

[2]it is sufficient to demand that multiplication only by nonnegative real scalar is defined, as done in [16]; however such generalization is not essential

Recall that opposite and negation are defined for $a = (A', A'') \in \mathcal{G}$ as:

$$\begin{aligned} -a &= -(A', A'') = (A'', A'), \\ (-1) * a &= (-1) * (A', A'') = ((-1) * A', (-1) * A''); \end{aligned}$$

the latter formula can be also written symbolically in the form:

$$\neg a = \neg(A', A'') = (\neg A', \neg A'').$$

The composition of "$-$" and "$\neg$" in $\mathcal{G}$ is called *dualization* (or *conjugation*) and is denoted by $a_- = -(\neg a) = \neg(-a)$. For the pair-wise presentation we obtain

$$a_- = (A', A'')_- = \neg(-(A', A'')) = \neg(A'', A') = (\neg A'', \neg A') = ((-1)*A'', (-1)*A').$$

We extend our symbolic notation as follows: for $a \in \mathcal{G}$ we write $a = a_+$; then for $\sigma \in \{+, -\}$ the element $a_\sigma \in \mathcal{G}$ is either a or a_- according to the binary value of σ.

The properties of negation, opposite and dualization in $\mathcal{G}$ are studied in [11]. It deserves to mention the following property. Let $\mathcal{G}_0$ be the image of $\mathcal{Q}_0$ under the embedding of $\mathcal{Q}$ into $\mathcal{G}$, that is: $\mathcal{G}_0 = \{(A, 0) \mid A \in \mathcal{Q}_0\}$. Then negation on $\mathcal{G}_0$ coincides with opposite, and dualization on $\mathcal{G}_0$ coincides with identity.

Proposition 11 *Let $(\mathcal{Q}, \mathcal{Q}_0, +, \mathbb{R}, *)$ be a quasilinear system, $(\mathcal{G}, \mathcal{G}_0, +, -, \neg)$ be the extension group according to Proposition 10, Part A, and multiplication by scalar "$*$" be defined on $\mathcal{G}$ by (29). Then,*

i) the system $(\mathcal{G}, \mathcal{G}_0, +, \mathbb{R}, *)$ is q-linear in the sense that "$*$" satisfies (14)–(16), that is for $a, b, c \in \mathcal{G}$, $\alpha, \beta, \gamma \in \mathbb{R}$: $\alpha*(\beta*c) = (\alpha\beta)*c$; $\gamma*(a+b) = \gamma*a+\gamma*b$; $1 * a = a$; together with the following distributivity relation:

$$(\alpha + \beta) * c_{\sigma(\alpha+\beta)} = \alpha * c_{\sigma(\alpha)} + \beta * c_{\sigma(\beta)}; \tag{32}$$

ii) inclusion in G defined by $(A, B) \subset (C, D) \Longleftrightarrow A + D \subset B + C$ is isotone with respect to addition and scalar multiplication, i. e. $a \subset b \Longleftrightarrow a + c \subset b + c$; $a \subset b \Longleftrightarrow \lambda * a \subset \lambda * b$, $\lambda \in \mathbb{R}$.

Proof. Relations (14)–(16) are obvious. To prove (32), note that it is equivalent to

$$(\alpha + \beta) * c = (\alpha * c + \beta * c_{\sigma(\alpha)\sigma(\beta)})_{\sigma(\alpha)\sigma(\alpha+\beta)}; \tag{33}$$

we shall prove (32) in this latter form. Substitute $c = (U, V) \in \mathcal{G}$ with $U, V \in \mathcal{Q}$. The right-hand side of (33) is:

$$r = (\alpha * (U, V) + \beta * (U, V)_{\sigma(\alpha)\sigma(\beta)})_{\sigma(\alpha)\sigma(\alpha+\beta)}.$$

If $\sigma(\alpha)\sigma(\beta) = +$ (and hence $\sigma(\alpha)\sigma(\alpha + \beta) = +$), using (17) we see that r is identical to the left-hand side:

$l = (\alpha+\beta)*(U,V) = ((\alpha+\beta)*U,\ (\alpha+\beta)*V)$.

Consider now the case $\sigma(\alpha)\sigma(\beta) = -$. The right-hand side becomes

$$\begin{aligned} r &= (\alpha*(U,V) + \beta*(U,V)_-)_{\sigma(\alpha)\sigma(\alpha+\beta)} \\ &= (\alpha*(U,V) + \beta*(\neg V, \neg U))_{\sigma(\alpha)\sigma(\alpha+\beta)} \\ &= ((\alpha*U, \alpha*V) + ((-\beta)*V, (-\beta)*U)_{\sigma(\alpha)\sigma(\alpha+\beta)} \\ &= (\alpha*U + (-\beta)*V,\ \alpha*V + (-\beta)*U)_{\sigma(\alpha)\sigma(\alpha+\beta)}. \end{aligned}$$

Now we have to consider a number of subcases. Consider, e. g. the subcase $\sigma(\alpha) = +$, $\sigma(\beta) = -$, $\sigma(\alpha+\beta) = +$ (in this subcase we have $\alpha \geq -\beta > 0$). Adding the zero term $(-\beta)*(U+V, U+V) = (0,0)$ to the left-hand side and using the quasidistributive law (17) we obtain:

$$\begin{aligned} l &= (\alpha+\beta)*(U,V) + (-\beta)*(U+V, U+V) \\ &= ((\alpha+\beta)*U, (\alpha+\beta)*V) + ((-\beta)*U + (-\beta)*V, (-\beta)*U + (-\beta)*V) \\ &= ((\alpha+\beta)*U + (-\beta)*U + (-\beta)*V,\ (\alpha+\beta)*V) + (-\beta)*V + (-\beta)*U) \\ &= (\alpha*U + (-\beta)*V,\ \alpha*V) + (-\beta)*U) = r. \end{aligned}$$

The rest of the cases are treated analogously. Part B about inclusion is also easily verified. □

Relation (32) (or (33)) are called *q-distributive law*. The q-distributive law can be also written in the form: $(\alpha+\beta)c = \alpha c_\lambda + \beta c_\mu$, with $\lambda = \sigma(\alpha)\sigma(\alpha+\beta)$, $\mu = \sigma(\beta)\sigma(\alpha+\beta)$.

The next proposition shows that the linear space from Proposition 10 is included in the q-linear space introduced in Proposition 11.

Proposition 12 *Let $(\mathcal{G}, \mathcal{G}_0, +, \mathbb{R}, *)$ be q-linear system in the sense of Proposition 11, i), and the operation "$\cdot$": $\mathbb{R} \times \mathcal{G} \longrightarrow \mathcal{G}$ be defined by*

$$\alpha \cdot c = \alpha * c_{\sigma(\alpha)}, \quad \alpha \in \mathbb{R},\ c \in \mathcal{G}. \tag{34}$$

Then $(\mathcal{G}, +, \mathbb{R}, \cdot)$ is a linear system.

Proof. Let us check that "$\cdot$" satisfies the axioms for linear multiplication.

1. Let us prove that $\alpha \cdot (\beta \cdot d) = (\alpha\beta) \cdot d$. Substitute $c = d_{\sigma(\beta)}$ in the relation $\alpha * (\beta * c) = (\alpha\beta) * c$ to obtain $\alpha * (\beta * d_{\sigma(\beta)}) = (\alpha\beta) * d_{\sigma(\beta)}$. Using (34) we have $\alpha * (\beta \cdot d) = (\alpha\beta) * d_{\sigma(\beta)}$. "Dualizing" by $\sigma(\alpha)$ we obtain $\alpha * (\beta \cdot d)_{\sigma(\alpha)} = (\alpha\beta) * d_{\sigma(\beta)\sigma(\alpha)} = (\alpha\beta) * d_{\sigma(\beta\alpha)}$, or $\alpha \cdot (\beta \cdot d) = (\alpha\beta) \cdot d$, for all $d \in \mathcal{G}$, $\alpha, \beta \in \mathbb{R}$.

2. To prove the relation $\gamma \cdot (a+b) = \gamma \cdot a + \gamma \cdot b$, substitute $a = c_{\sigma(\gamma)}$, $b = d_{\sigma(\gamma)}$ in $\gamma * (a+b) = \gamma * a + \gamma * b$. We obtain $\gamma * (c_{\sigma(\gamma)} + d_{\sigma(\gamma)}) = \gamma * c_{\sigma(\gamma)} + \gamma * d_{\sigma(\gamma)}$, or $\gamma * (c+d)_{\sigma(\gamma)} = \gamma * c_{\sigma(\gamma)} + \gamma * d_{\sigma(\gamma)}$. This implies that $\gamma \cdot (c+d) = \gamma \cdot c + \gamma \cdot d$, for all $c, d \in \mathcal{G}$, $\gamma \in \mathbb{R}$.

The relations $1 \cdot a = a$, $(\alpha+\beta) \cdot c = \alpha \cdot c + \beta \cdot c$ and $(-1) \cdot a + a = 0$ can be proved similarly. □

We proved that the system $(\mathcal{G}, +, \mathbb{R}, \cdot)$ with multiplication by scalar "$\cdot$" defined by (34) is a linear system, hence, "$\cdot$" is the linear multiplication by scalar (31) appearing in Proposition 10. From (34) we see that the operation "$\cdot$" is involved in the q-linear system $(\mathcal{G}, \mathcal{G}_0, +, \mathbb{R}, *)$ as composite operation — therefore the latter can be written in the form $(\mathcal{G}, \mathcal{G}_0, +, \mathbb{R}, *, \cdot)$. Note that the inverse is not true — Raadström's vector space $(\mathcal{G}, +, \mathbb{R}, \cdot)$ does not involve the natural extension of the multiplication by real scalar "$*$".

The q-linear system $(\mathcal{G}, \mathcal{G}_0, +, \mathbb{R}, *, \subset)$ considered in Proposition 11 can be also endowed with metric $\delta = \delta(x, y)$ and norm $\parallel x \parallel = \delta(x, 0)$ in the same manner as done in Proposition 10 Part C. Due to Proposition 12 the linear multiplication (31) is incorporated in the q-linear system, so that the function $x + (-1) \cdot y = x - y = x \neg y_-$ can be constructed in $\mathcal{G}$ (here "$-$" is the opposite and "$\neg$" is the negation in $\mathcal{G}$. We have $\delta(x, y) = \parallel x - y \parallel = \parallel x \neg y_- \parallel$, and $\parallel x \parallel = \delta(x, 0)$. The system $(\mathcal{G}, \mathcal{G}_0, +, -, \mathbb{R}, *, \subset, \parallel \cdot \parallel)$ thus obtained is called a *q-linear space*.

Proposition 11 is a generalization of Raadström's embedding theorem [16] in the directions: a) no restriction for the signs of the scalar multipliers in the second distributive law (that is in the quasi- and q-distributive laws) are required (leading to embedding of cones in Raadström case), and b) an extension of the inclusion relation is given; c) the q-linear space involves the linear space from Proposition 10. Clearly, (29) isomorphically extends multiplication by scalar from $\mathcal{Q}$ into $\mathcal{G}$; briefly, Proposition 11 says that a quasilinear system can be isomorphically embedded into a q-linear system.

Example 1. It has been shown by H. Raadstöm that the system $(\mathcal{K}(\mathbb{E}), +, \mathbb{R}, *)$ satisfies the conditions of Proposition 10 and therefore can be extended up to a normed vector space. Note that the extention of "$*$" is not a isomorphic, and one can only speak of embedding of $\mathcal{K}$ as a convex cone. According to Proposition 11 and Proposition 12 the system $(\mathcal{K}(\mathbb{E}), +, \mathbb{R}, *)$ can be isomorphically embedded in a normed q-linear space, which incorporates a normed linear space.

Example 2. Another example of quasilinear and q-linear space can be constructed from the set $\mathcal{K}_S$ of symmetric bodied considered in Section 3. The subgroup of invertible elements of $\mathcal{K}_S$ is the trivial group $\{0\}$. The quasilinear system of symmetric elements is $(\mathcal{K}_S, 0, +, \mathbb{R}, *)$. Due to $\neg B = (-1) * B = B$ it is easy to check that $\alpha * B = |\alpha| * B$ for $B \in \mathcal{K}_S$. Using (13), this implies $\alpha * B = \{\alpha x | x \in B\} = \{|\alpha| x \mid x \in B\}$. By Proposition 11 the quasilinear system $(\mathcal{K}_S, 0, +, \mathbb{R}, *)$ can be embedded in a q-linear system, which is a subsystem of the q-linear system of general convex bodies.

7 Conclusions

Algebraic properties of convex bodies with respect to Minkowski operations for addition and multiplication by real scalar are studied. It is demonstrated that the quasilinear system of convex bodies can be isomorphically embedded into a q-linear system, having group properties w. r. t. addition. The quasidistributive law induces in the q-linear system a simple distributivity relation, called q-distributive law. A q-linear system has much algebraic structure and is rather close to a linear system and differs from the latter by:

i) existence of two new automorphic operators — "negation" and "dualization" — in addition to the familiar automorphism "opposite" (and, of course, identity);

ii) the distributivity relation (q-distributive law) resambles the usual linear distributivity law with the difference that the operator dualization is involved.

The q-linear system is endowed with metric, norm and inclusion; it has been shown that the q-linear space involves the vector space as discussed by H. Raadström.

Some rules for computation in a q-linear space are given in [9]–[11]. In a forthcoming paper our main results will be formulated in terms of support functions, cf. [2].

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References

[1] Alefeld G., J. Herzberger, Introduction to Interval Computation, Academic Press, 1983.

[2] Baier, R., E. Farhi, Directed Sets and Differences of Convex Compact Sets. Proc. 18th IFIP TC7 Conference on System Modelling and Optimization, Detroit, July 1997.

[3] Clifford, A. H., G. B. Preston, The Algebraic Theory of Semigroups, I, AMS, 1964.

[4] Hadwiger H., Minkowskische Addition und Subtraktion beliebiger Punktmengen und die Theoreme von Rirhard Schmidt. Math. Z., 53 (1953), Heft 3, 210–218.

[5] Hadwiger H., Vorlesungen über Inhalt, Oberfläche und Isoperimetrie. Springer, Berlin, 1957.

[6] Kaucher, E., Algebraische Erweiterungen der Intervallrechnung unter Erhaltung der Ordnungs- und Verbandstrukturen, Computing Suppl., 1 (1977), 65–79.

[7] Kaucher, E., Interval Analysis in the Extended Interval Space $I\mathbb{R}$, Computing Suppl. 2, 33–49 (1980).

[8] Kurosh, A. G., Lectures on General Algebra, Chelsea, 1963.

[9] Markov, S., Isomorphic Embedding of Abstract Interval Systems, *Reliable Computing*, 1997, Vol. 3, 3, 199–207.

[10] Markov, S., On the Algebra of Intervals and Convex Bodies, J.UCS, 4, 1, 34–47 (1998).

[11] Markov, S., On the Properties of Convex Bodies and Related Algebraic Systems, To appear in Ann. Univ. Sofia, Fac. Math. Inf., v. 91, livre 1.

[12] Matheron, G., Random Sets and Integral Geometry, Wiley, 1975.

[13] Ohmann, D., Eine vollständiges Ungleichungssystem für die Minkowskische Summe und Differenz, Comment. Math. Helv., 27, 151–156 (1953).

[14] Ohmann, D., Ungleichungen für die Minkowskische Summe und Differenz konvexer Körper, Comment. Math. Helvet., 30, 1956, 297–307.

[15] Popov, A., A Relation between Morphological and Interval Operations, Reliable Computing, 4, 2, 1998, 167–178.

[16] Rådström, H., An Embedding Theorem for Spaces of Convex Sets, Proc. Amer. Math. Soc. 3 (1952), 165–169.

[17] Rockafellar, R. T., Convex Analysis, Princeton, 1970.

[18] Schneider, R., Convex Bodies: The Brunn-Minkowski Theory. Cambridge Univ. Press, 1993.