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Wrapping Effect and Wrapping Function

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Abstract. We study the wrapping effect associated with validated interval methods for numerical solution of the initial value problem for ordinary differential equations by introducing a new concept of wrapping function. The wrapping function is proved to be the limit of the enclosures of the solution produced by methods of certain type. There is no wrapping effect if and only if the wrapping function equals the optimal interval enclosure of the solution.

1. Introduction

We discuss the wrapping effect associated with validated numerical methods for the initial value problem

$$\dot{x} = f(t,x), \tag{1.1}$$

$$x(t_0) = x^0 \in X^0, \tag{1.2}$$

where $t \in [t_0, \bar{t}]$ is an interval on the real line $\mathbb{R}$, $x^0 \in \mathbb{R}^n$, $D \subset \mathbb{R}^n$ is an open set, $f : [t_0, \bar{t}] \times D \to \mathbb{R}^n$ and

$$X^0 = ([\underline{x}_1^0, \bar{x}_1^0], [\underline{x}_2^0, \bar{x}_2^0], \ldots, [\underline{x}_n^0, \bar{x}_n^0])^T$$

is an n-dimensional interval vector, $X^0 \subset D$. We assume that a solution is sought in some interval $[t_0, \bar{t}]$. A validated interval numerical method either gives a special notice, or produces an n-dimensional interval function $S(h;t)$, $t \in [t_0, \bar{t}]$ with the assurance that for every $x^0 \in X^0$ a unique solution $x(t_0, x^0; t)$ of (1.1)–(1.2) exists in $[t_0, \bar{t}]$ and $x(t_0, x^0; t) \in S(h;t)$, $t \in [t_0, \bar{t}]$. The step size $h > 0$ is a parameter of the method. We consider methods that generate an interval enclosure $S(h;t)$ using a mesh $\{t_0, t_1, \ldots, t_p = \bar{t}\}$. To simplify the presentation we consider a uniform mesh with step size h; however our results are true for a non-uniform mesh as well.

An interval vector $Y = ([\underline{y}_1, \bar{y}_1], [\underline{y}_2, \bar{y}_2], \ldots, [\underline{y}_n, \bar{y}_n])^T$ is defined as the set

$$Y = \{(y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n : y_i \in [\underline{y}_i, \bar{y}_i],\ i = 1, 2 \ldots, n\}$$

and it is also represented in the form

$$Y = [\underline{y}, \overline{y}] = \{y \in \mathbb{R}^n : \underline{y} \leq y \leq \overline{y}\}$$

where $\underline{y} = (\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_n)^T$, $\overline{y} = (\overline{y}_1, \overline{y}_2, \ldots, \overline{y}_n)^T$ and "$\leq$" is a partial ordering in R^n defined by

$$(y_1, y_2, \ldots, y_n)^T \leq (z_1, z_2, \ldots, z_n)^T \iff y_i \leq z_i, \quad i = 1, 2, \ldots, n.$$

Obviously, the symbol $[\underline{y}, \overline{y}]$ denotes an interval vector iff $\underline{y} \leq \overline{y}$. The space of n-dimensional interval vectors is denoted by $I(\mathbb{R})^n$. The distance between two intervals $Y, Z \in I(\mathbb{R})^n$ is defined as the Hausdorff distance between Y and Z considered as subsets of $\mathbb{R}^n$ and is denoted by $\wp(Y, Z)$.

The set-valued function $x(t_0, X^0; t) = \{x(t_0, x^0; t) : x^0 \in X^0\}$, $t \in [t_0, \bar{t}]$ (which is not necessarily an interval), is called a *solution* to problem (1.1)–(1.2). For every $t \in [t_0, \bar{t}]$ we denote the optimal (tightest) interval containing the set $x(t_0, X^0; t)$ by $[x(t_0, X^0; t)]$. The interval function

$$[x(t_0, X^0; \cdot)] : [t_0, \bar{t}] \longrightarrow I(\mathbb{R})^n$$

is called *optimal interval enclosure of the solution* (in $[t_0, \bar{t}]$). A validated interval method produces interval functions $S(h; t)$ such that $[x(t_0, X^0; t)] \subset S(h; t)$. Naturally, convergence of the form

$$\lim_{h \to 0} S(h; t) = [x(t_0, X^0; t)]$$

is desirable. However, in some cases such convergence is not observed. This is demonstrated in the following example.

EXAMPLE 1. Consider the problem

$$\begin{aligned}
\dot{x}_1 &= -2x_1, & x_1(0) &= x_1^0 \in X_1^0 = 1 + [-\varepsilon_1, \varepsilon_1], \\
\dot{x}_2 &= 2x_1 - x_3, & x_2(0) &= x_2^0 \in X_2^0 = 1 + [-\varepsilon_2, \varepsilon_2], \\
\dot{x}_3 &= 2x_1 - x_2, & x_3(0) &= x_3^0 \in X_3^0 = 1 + [-\varepsilon_3, \varepsilon_3]
\end{aligned} \tag{1.3}$$

in the interval $[0, 1]$. We apply a method based on the Taylor series of the solution with local approximation error $O(h^5)$. In every interval $[t_k, t_{k+1}]$ the already computed enclosure $S(h; t_k)$ is considered as an initial condition and we have

$$\wp\Big(S(h; t), \big[x\big(t_k, S(h, t_k); t\big)\big]\Big) = O(h^5), \quad t \in [t_k, t_{k+1}].$$

For

$$\varepsilon_1 = 0.2, \quad \varepsilon_2 = \varepsilon_3 = 0.05 \tag{1.4}$$

the optimal interval enclosure and enclosures computed for three values of h are visualized in Figure 1 (the enclosures for x_2 and x_3 coincide). While the numerically

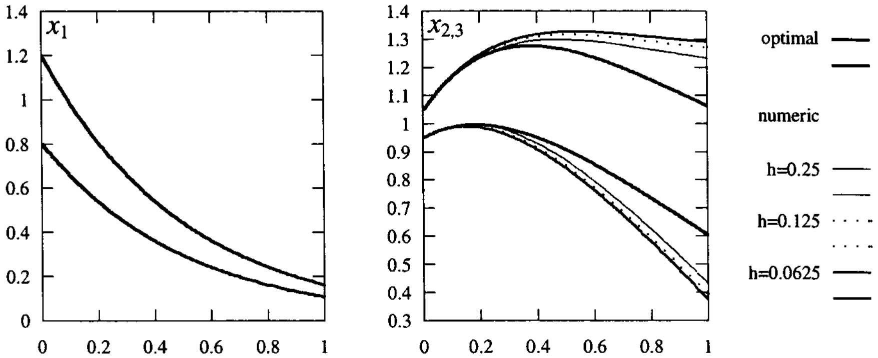

Figure 1. *Problem (1.3) with* $\varepsilon_1 = 0.2$, $\varepsilon_2 = \varepsilon_3 = 0.05$. *Optimal enclosure and enclosures computed numerically for various step sizes h.*

computed enclosures for x_1 are visually indistinguishable from the optimal one, the computed enclosures for x_2 and x_3 clearly diverge from the optimal enclosure when reducing the step size. Increasing the order of local approximation is of no help.

However, when the same method is applied to Problem (1.3) with

$$\varepsilon_1 = 0, \quad \varepsilon_2 = \varepsilon_3 = 0.05 \tag{1.5}$$

we obtain a very good approximation of the optimal interval enclosure in all three variables x_1, x_2, and x_3. The numerical results are graphically represented in Figure 2. At the top part of Figure 2 the optimal enclosure and the enclosures computed for various values of h are plotted. Since the computed enclosures are very close to the optimal enclosure they are visually indistinguishable from the optimal one. In order to see their accuracy the error functions

$$\wp\left(S_i(h;t),\ [x_i(t_0, X^0; t)]\right), \quad i = 1, 2, 3$$

are plotted on a logarithmic scale at the bottom part of the figure. A rate of convergence, consistent with the expected rate of global convergence $O(h^4)$ is observed.

The divergence of the computed interval enclosures away from the optimal enclosure when $h \to 0$ observed for x_2 and x_3 on Figure 1 is due to the wrapping effect. A detailed explanation of the wrapping effect will be given below; here we can say roughly that the latter is manifested as a divergence of the computed enclosures away from the optimal enclosure when $h \to 0$ irrespectively of the order of local approximation. We can see from the numerical experiments with Example 1 that there are problems (like (1.3)–(1.5)) where the wrapping effect does not appear at all and the computed enclosures behave in a "regular" way, i.e. converge to the optimal enclosure when $h \to 0$.

We shall explain the wrapping effect using L. Jackson's [3] "propagate and wrap" approach. Suppose that we can compute the optimal interval enclosure of the solution in any interval $[t_k, t_{k+1}]$. Then interval enclosures can be computed by

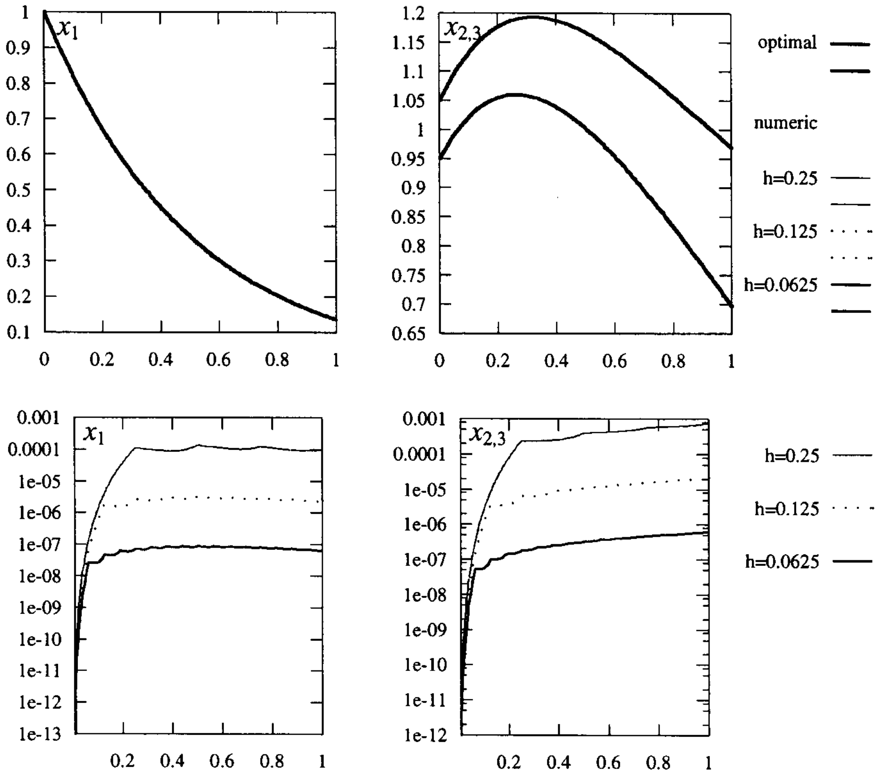

Figure 2. *Problem (1.3) with* $\varepsilon_1 = 0$, $\varepsilon_2 = \varepsilon_3 = 0.05$. *Optimal enclosure and enclosures computed numerically for various step sizes h (top) and errors of the computed enclosures on a logarithmic scale (bottom).*

the following procedure which we call Idealized Propagate and Wrap Algorithm or shortly IPWA:

$$\begin{aligned} \mathcal{S}(h;t_0) &= X^0, \\ \mathcal{S}(h;t) &= \left[x\left(t_k, \mathcal{S}(h;t_k);t\right)\right], \quad t \in [t_k, t_{k+1}], \ k = 0,1,\ldots,n-1. \end{aligned} \tag{1.6}$$

This method has no local error but does not always produce the optimal enclosure of the solution in the interval $[t_0, \bar{t}]$. The solution at t_1 is the set $x(t_0, X^0; t_1)$ which is not necessarily an n-dimensional interval. It is "wrapped" by an interval $\mathcal{S}(h;t_1) = [x(t_0, X^0; t_1)]$ possibly including in this way extra points called *wrapping excess*. In the interval $[t_1, t_2]$ all solutions starting from the points of $\mathcal{S}(h;t_1)$ are propagated (including the wrapping excess) and the set $x(t_1, \mathcal{S}(h;t_1); t_2)$ is again wrapped by an interval $\mathcal{S}(h;t_2) = [x(t_1, \mathcal{S}(h;t_1); t_2)]$ with certain wrapping excess and so on. The accumulated wrapping excess at the points of the mesh is what causes in some cases the blowing of the enclosures as observed for variables $x_{2,3}$ on Figure 1 and referred to as *wrapping effect*. In other cases despite the wrapping excess the computed enclosures may converge to the optimal one (cf. Figure 2), i.e. there is no wrapping effect.

Problems associated with the wrapping excess at the points of the mesh are considered in [7] where coordinate transformations are proposed to eliminate the wrapping effect. A large number of papers on validated (interval) methods for ordinary differential equations deal with the wrapping effect. Some major developments in the area are marked in [1], [2], [4], [5], [9]; see also [8] for a recent survey.

A well known case of problems with no wrapping effect is when the function f in (1.1) is quasi-isotone.

DEFINITION. A function $f = (f_1, f_2, \dots, f_n)^T : [t_0, \bar{t}] \times D \to \mathbb{R}^n$ is called *quasi-isotone* on x if $f_i = f_i(t, x_1, x_2, \dots, x_n)$ is non-decreasing with respect to all $x_j, j \neq i$ for every $i = 1, \dots, n$.

There is also no wrapping effect when the initial condition is a point $X^0 = x^0 \in \mathbb{R}$. Problem (1.3)–(1.5) presents another case where no wrapping effect exists.

In this paper we study the wrapping effect by introducing a new concept of wrapping function. In all cases of wrapping effect convergence of the computed enclosures (although not to the optimal enclosure) is observed (see the graphs for $x_{2,3}$ in Figure 1). Then, if the enclosures do not converge to the optimal one what do they converge to? We shall prove that the enclosures computed by any method of a certain type (which will be specified in the next section) converge to a "wrapping function" and that no wrapping effect exists when the wrapping function equals the optimal enclosure.

2. The Wrapping Function

We did not make any assumptions for existence and uniqueness of the solution of Problem (1.1)–(1.2) because a validated method either gives a special notice or produces guaranteed bounds for the solution and simultaneously verifies that a unique solution exists [8]. However, in proving (a priori) convergence we need to make such assumptions. To simplify the proof of the theorems we assume that in the region $[t_0, \bar{t}] \times D$ the function f is

i) bounded: $|f_i(t,x)| \leq m_i \in \mathbb{R},\ m = (m_1, m_2, \dots, m_n)^T \in \mathbb{R}^n$;

ii) continuous on t; (2.1)

iii) Lipschitzian on x:

$$|f_i(t,y) - f_i(t,z)| \leq \sum_{j=1}^{n} \lambda_{ij} |y_j - z_j|, \quad \lambda_{ij} \in \mathbb{R},\ i,j = 1, \dots, n.$$

However, all proofs can be carried out under more general assumptions providing for existence and uniqueness of a solution $x(t_0, x^0; t)$ in a weaker sense leading to a continuous function satisfying $x(t) = x^0 + \int_{t_0}^{t} f(\theta, x(\theta))\, \mathrm{d}\theta$, $t \in [t_0, \bar{t}]$.

We also assume that all solutions $x(t_0, x^0; t)$, $x^0 \in X^0$, are defined in the whole interval $[t_0, \bar{t}]$.

DEFINITION. A function $X : [t_0, \bar{t}] \to I(\mathbb{R})^n$ is said to satisfy a *wrapping property with respect to equation* (1.1) *at the point* $\theta \in [t_0, \bar{t})$ if for every $u \in X(\theta)$ there exists a unique solution $x(\theta, u; t)$ on the interval $[\theta, \bar{t}]$ satisfying (1.1) with $x(\theta) = u$ and $x(\theta, X(\theta); t) \subset X(t)$, $t \in [\theta, \bar{t}]$.

We consider methods producing enclosures $S(h; t)$ satisfying the wrapping property at the points of the mesh. We refer to these methods as *methods of propagate and wrap type*. The Idealized Propagate and Wrap Algorithm (IPWA) produces the tightest enclosure $\boldsymbol{S}(h; t)$.

Since the enclosures produced by methods of propagate and wrap type satisfy the wrapping property at every point of the mesh we may expect that the limit of such interval functions when $h \to 0$ satisfies the wrapping property at every point of the interval $[t_0, \bar{t}]$. Therefore we define a wrapping function as follows:

DEFINITION. A function $X : [t_0, \bar{t}] \to I(\mathbb{R})^n$ is called *wrapping function* for the problem (1.1)–(1.2) if:

i) $X(t_0) = X^0$;

ii) X satisfies the wrapping property at every point of the interval $[t_0, \bar{t})$;

iii) $X(t) \subset Y(t)$, $t \in [t_0, \bar{t}]$ holds for every function $Y : [t_0, \bar{t}] \to I(\mathbb{R})^n$ satisfying i) and ii).

Condition iii) means that the wrapping function is the optimal function satisfying properties i) and ii). Roughly speaking, the wrapping function is the tightest interval function which wraps all excess points and the induced solutions through these points.

The wrapping function of Problem (1.1)–(1.2) is unique. Indeed, if Y and Z are wrapping functions then the function X defined by $X(t) = Y(t) \cap Z(t)$, $t \in [t_0, \bar{t}]$ satisfies the conditions i) and ii) of the definition and $X(t) \subset Y(t)$, $X(t) \subset Z(t)$, $t \in [t_0, \bar{t}]$. Moreover, the converse inclusions hold according to iii). Therefore $X(t) = Y(t) = Z(t)$. We denote the wrapping function of Problem (1.1)–(1.2) by $\widehat{X}$. To prove its existence and further properties the wrapping function $\widehat{X}$ will be represented as a solution to an initial value problem involving the interval extension of the function f defined as follows.

Let $I\!D = \{X \in I(\mathbb{R})^n : X \subset D\}$. A function $f^* : [t_0, \bar{t}] \times ID \to I(\mathbb{R})^n$ defined by

$$\begin{aligned} f^*(t, X) &= \left(f_1^*(t, X), f_2^*(t, X), \dots, f_n^*(t, X)\right)^T, \\ f_i^*(t, X) &= [\underline{f}_i(t, X), \overline{f}_i(t, X)] = \left[\inf_{x \in X} f_i(t, x), \sup_{x \in X} f_i(t, x)\right], \quad i = 1, \dots, n \end{aligned}$$

is called *interval extension* of f (with respect to x). Since the interval extension f^* equals f when $X = x \in D$ the use of the same notation does not cause confusion, so that we denote further f^* again by f.

In $I(\mathbb{R})^n$ we define the quantities width $w(\cdot)$ and magnitude $|\cdot|$ for $Y \in I(\mathbb{R})^n$ respectively by

$$w(Y) = (\bar{y}_1 - \underline{y}_1, \bar{y}_2 - \underline{y}_2, \ldots, \bar{y}_n - \underline{y}_n)^T,$$
$$|Y| = \left(\max\{|\underline{y}_1|, |\bar{y}_1|\}, \max\{|\underline{y}_2|, |\bar{y}_2|\}, \ldots, \max\{|\underline{y}_n|, |\bar{y}_n|\}\right)^T.$$

In $I(\mathbb{R})^n$ we use the interval operations $Y + Z$, αY, defined for $Y, Z \in I(\mathbb{R})^n$, $\alpha \in \mathbb{R}$, by

$$Y + Z = (Y_1 + Z_1,\ Y_2 + Z_2, \ldots, Y_n + Z_n)^T, \quad Y_i + Z_i = [\underline{y}_i + \underline{z}_i,\ \bar{y}_i + \bar{z}_i];$$
$$\alpha Y = (\alpha Y_1, \alpha Y_2, \ldots, \alpha Y_n)^T,$$
$$\alpha Y_i = [\min\{\alpha \underline{y}_i, \alpha \bar{y}_i\},\ \max\{\alpha \underline{y}_i, \alpha \bar{y}_i\}], \quad i = 1, \ldots, n,$$

and the inner subtraction of intervals $Y -^- Z$, $Y, Z \in I(\mathbb{R})^n$, see e.g. [6], by

$$Y -^- Z = (Y_1 -^- Z_1,\ Y_2 -^- Z_2, \ldots, Y_n -^- Z_n)^T,$$
$$Y_i -^- Z_i = [\min\{\underline{y}_i - \underline{z}_i,\ \bar{y}_i - \bar{z}_i\},\ \max\{\underline{y}_i - \underline{z}_i,\ \bar{y}_i - \bar{z}_i\}].$$

The distance between intervals $Y, Z \in I(\mathbb{R})^n$ can be represented in the form

$$\wp(Y, Z) = \max_{i = 1, \ldots, n} |Y_i -^- Z_i| = \| |Y -^- Z| \|$$

where $\|\cdot\| : \mathbb{R}^n \mapsto \mathbb{R}$ is the maximum norm in $\mathbb{R}^n$ defined by

$$\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}, \quad x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

Conditions (2.1) on f imply corresponding properties of the interval extension $f(t, X) = [\underline{f}(t, X), \bar{f}(t, X)]$ of $f(t, x)$ with respect to x. In the region $I\!D$ the interval extension f is

i) bounded: $|f_i(t, X)| \le m_i \in \mathbb{R}$, $m = (m_1, m_2, \ldots, m_n)^T \in \mathbb{R}^n$;

ii) continuous on t (which means that $\underline{f}$ and $\bar{f}$ are continuous on t); (2.2)

iii) Lipschitzian on X in the form:

$$|f(t, Y) -^- f(t, Z)| \le \Lambda |Y -^- Z| \quad \text{where} \quad \Lambda = (\lambda_{ij}) \in \mathbb{R}^{n \times n}.$$

We shall also use the following notation. Let $Y \in I(\mathbb{R})^n$, then

$$\underline{Y}^i = (Y_1, \ldots, Y_{i-1}, \underline{y}_i, Y_{i+1}, \ldots, Y_n)^T,$$
$$\bar{Y}^i = (Y_1, \ldots, Y_{i-1}, \bar{y}_i, Y_{i+1}, \ldots, Y_n)^T.$$

Let $X : [t_0, \bar{t}] \to I\!D$ be an interval function. The interval operator $\mathcal{L}$ is defined by

$$\mathcal{L}X(t) = \begin{pmatrix} [\dot{\underline{x}}_1(t) - \underline{f}_1(t, \underline{X}^1(t)),\ \dot{\bar{x}}_1(t) - \bar{f}_1(t, \overline{X}^1(t))] \\ [\dot{\underline{x}}_2(t) - \underline{f}_2(t, \underline{X}^2(t)),\ \dot{\bar{x}}_2(t) - \bar{f}_2(t, \overline{X}^2(t))] \\ \cdots\cdots\cdots\cdots \\ [\dot{\underline{x}}_n(t) - \underline{f}_n(t, \underline{X}^n(t)),\ \dot{\bar{x}}_n(t) - \bar{f}_n(t, \overline{X}^n(t))] \end{pmatrix}, \quad t \in [t_0, \bar{t}].$$

We consider the following initial value problem

$$\mathcal{L}X = 0, \tag{2.3}$$

$$X(t_0) = X^0. \tag{2.4}$$

From (2.2) it follows that this problem has a unique solution in some interval $[t_0, \sigma]$. For simplicity we assume $\sigma = \bar{t}$.

THEOREM 2.1.

a) The solution of problem (2.3)–(2.4) is the wrapping function of Problem (1.1)–(1.2);

b) The interval enclosures $\mathcal{S}(h;t)$ produced by IPWA converge to the wrapping function of Problem (1.1)–(1.2) when $h \to 0$ i.e. $\lim_{h\to 0} \mathcal{S}(h;t) = \widehat{X}(t)$.

Proof. Denote the solution of problem (2.3)–(2.4) by

$$X(t_0, X^0; t) = [\underline{x}(t_0, X^0; t),\ \bar{x}(t_0, X^0; t)].$$

We shall first prove the following inclusions

$$\mathcal{S}(h;t) \subset \widehat{X}(t) \subset X(t_0, X^0; t), \quad t \in [t_0, \bar{t}],\ h > 0. \tag{2.5}$$

The first inclusion follows straightforward from the definition of wrapping function. We shall use some monotone properties of the interval extension of f to show the second inclusion. Every component $f_i(t,X) = [\underline{f}_i(t,X), \bar{f}_i(t,X)]$, $i = 1, \ldots, n$, of $f(t,X)$ satisfies

$$\underline{f}_i(t,X) = \underline{f}_i(t, [\underline{x}_1, \bar{x}_1], [\underline{x}_2, \bar{x}_2], \ldots, [\underline{x}_n, \bar{x}_n]) \text{ is } \left\{ \begin{array}{l} \text{non-decreasing on } \underline{x}_i \\ \text{non-increasing on } \bar{x}_i \end{array} \right\},$$

$$\bar{f}_i(t,X) = \bar{f}_i(t, [\underline{x}_1, \bar{x}_1], [\underline{x}_2, \bar{x}_2], \ldots, [\underline{x}_n, \bar{x}_n]) \text{ is } \left\{ \begin{array}{l} \text{non-increasing on } \underline{x}_i \\ \text{non-decreasing on } \bar{x}_i \end{array} \right\}.$$

Therefore the function $g = (g_1, g_2, \ldots, g_{2n})$ defined in the region

$$\{(t,y) : t \in [t_0, \bar{t}],\ y \in D \times D,\ y_j + y_{n+j} \geq 0,\ j = 1, \ldots, n\}$$

by

$$\begin{aligned} g_i(t, y_1, y_2, \ldots, y_{2n}) &= -\underline{f}_i(t, Y_1, \ldots, Y_{i-1}, -y_i, Y_{i+1}, \ldots, Y_n), \quad i = 1, \ldots, n, \\ g_{n+i}(t, y_1, y_2, \ldots, y_{2n}) &= \bar{f}_i(t, Y_1, \ldots, Y_{i-1}, y_{n+i}, Y_{i+1}, \ldots, Y_n), \quad i = 1, \ldots, n \end{aligned}$$

where $Y_j = [-y_j, y_{n+j}], j = 1, \ldots, n$, is quasi-isotone.

Consider the equation

$$\dot{y} = g(t, y). \tag{2.6}$$

A well known property of equations with a quasi-isotone right-hand side is that if $y(t)$ and $z(t)$ are two solutions of (2.6) such that $y(\theta) \le z(\theta)$ for some $\theta \in [t_0, \bar{t})$ then $y(t) \le z(t),\ t \in [\theta, \bar{t}]$ [9].

Let $\theta \in [t_0, \bar{t}]$ and let $x(\theta, u; t)$ be any solution of equation (1.1) satisfying $x(\theta) = u \in X(t_0, X^0; \theta)$. It is easy to see that the two $2n$-dimensional functions $(-x(\theta, u; t),\ x(\theta, u; t))$ and $(-\underline{x}(t_0, X^0; t),\ \overline{x}(t_0, X^0; t))$ are solutions of equation (2.6). At the point θ we have

$$\big(-x(\theta, u; \theta),\ x(\theta, u; \theta)\big) = (-u, u) \le \big(-\underline{x}(t_0, X^0; \theta),\ \overline{x}(t_0, X^0; \theta)\big).$$

Therefore $(-x(\theta, u; t),\ x(\theta, u; t)) \le (-\underline{x}(t_0, X^0; t),\ \overline{x}(t_0, X^0; t)),\ t \in [\theta, \bar{t}]$, which implies that $\underline{x}(t_0, X^0; t) \le x(\theta, u; t) \le \overline{x}(t_0, X^0; t),\ t \in [\theta, \bar{t}]$. Hence

$$x(\theta, u; t) \in [\underline{x}(t_0, X^0; t),\ \overline{x}(t_0, X^0; t)] = X(t_0, X^0; t), \quad t \in [\theta, \bar{t}].$$

Since the last inclusion is true for every $u \in X(t_0, X^0; \theta)$ and $\theta \in [t_0, \bar{t}]$ it follows that $X(t_0, X^0; t)$ satisfies the wrapping property at every point of the interval $[t_0, \bar{t})$. But the wrapping function is the optimal function that satisfies $X(t_0) = X^0$ and the wrapping property at every point of $[t_0, \bar{t})$. Therefore $\widehat{X}(t) \subset X(t_0, X^0; t),\ t \in [t_0, \bar{t}]$. This proves the inclusion (2.5).

Now we shall prove that $\lim_{h \to 0} \boldsymbol{S}(h; t) = X(t_0, X^0; t)$. This together with (2.5) implies both a) and b) in the theorem.

Let $[t_k, t_{k+1}]$ be an arbitrary subinterval and let $t \in [t_k, t_{k+1}]$. Then $\boldsymbol{S}(h; t)$ is defined by $\boldsymbol{S}(h; t) = [\underline{s}(h; t), \overline{s}(h; t)] = [x(t_k, \boldsymbol{S}(h; t_k); t)]$ where

$$\underline{s}_i(h; t) = \min_{u \in \boldsymbol{S}(h; t_k)} x_i(t_k, u; t), \quad \overline{s}_i(h; t) = \max_{u \in \boldsymbol{S}(h; t_k)} x_i(t_k, u; t), \quad i = 1, \ldots, n.$$

Using

$$|x(t_k, u; t) - u| = \left| \int_{t_k}^{t} f\big(\theta, x(t_k, u; \theta)\big)\, \mathrm{d}\theta \right| \le (t - t_k)m$$

it can be shown that

$$|\boldsymbol{S}(h; t) -^{-} \boldsymbol{S}(h; t_k)| \le (t - t_k)m. \tag{2.7}$$

Every solution $x(t_k, u; t)$ can be represented in the form

$$\begin{aligned} x(t_k, u; t) &= u + \int_{t_k}^{t} f\big(\theta, x(t_k, u; \theta)\big)\, \mathrm{d}\theta \\ &= u + \int_{t_k}^{t} f(\theta, u)\, \mathrm{d}\theta + \int_{t_k}^{t} \Big(f\big(\theta, x(t_k, u; \theta)\big) - f(\theta, u)\Big)\, \mathrm{d}\theta \\ &= \varphi(u) + \varepsilon \end{aligned}$$

where $\varphi(u) = u + \int_{t_k}^{t} f(\theta, u)\,\mathrm{d}\theta$ and

$$\begin{aligned} |\varepsilon| &= \left| \int_{t_k}^{t} \Big(f(\theta, x(t_k, u; \theta)) - f(\theta, u) \Big)\,\mathrm{d}\theta \right| \;\le\; \int_{t_k}^{t} \Lambda |x(t_k, u; \theta) - u|\,\mathrm{d}\theta \\ &\le \int_{t_k}^{t} \Lambda(\theta - t_k) m\,\mathrm{d}\theta \;=\; \frac{1}{2}(t - t_k)^2 \Lambda m \;\le\; \frac{1}{2} h^2 \Lambda m. \end{aligned}$$

Therefore for every $i = 1, 2, \dots, n$ we have

$$\varphi_i(u) - \frac{1}{2} h^2 \Lambda_{i*} m \le x_i(t_k, u; t) \le \varphi_i(u) + \frac{1}{2} h^2 \Lambda_{i*} m,$$

where Λ_{i*} is the i-th row of matrix Λ. Taking the maximum over $u \in \mathcal{S}(h; \theta)$ of every part in the above inequality we obtain

$$\overline{\varphi}_i(\mathcal{S}(h; t_k)) - \frac{1}{2} h^2 \Lambda_{i*} m \le \overline{s}_i(h; t) \le \overline{\varphi}_i(\mathcal{S}(h; t_k)) + \frac{1}{2} h^2 \Lambda_{i*} m,$$

which can also be written in the form

$$|\overline{s}_i(h; t) - \overline{\varphi}_i(\mathcal{S}(h; t_k))| \le \frac{1}{2} h^2 \Lambda_{i*} m. \tag{2.8}$$

Let us note that for sufficiently small h the function $\varphi = \varphi(u)$ is such that φ_i is non-decreasing on u_i, $i = 1, \dots, n$. Hence, for the interval extension of φ at $\mathcal{S}(h; t_k)$ we have

$$\begin{aligned} \overline{\varphi}_i(\mathcal{S}(h; t_k)) &= \max_{u \in \mathcal{S}(h; t_k)} \left\{ u_i + \int_{t_k}^{t} f_i(\theta, u)\,\mathrm{d}\theta \right\} \\ &= \overline{s}_i(h; t_k) + \int_{t_k}^{t} f_i(\theta, \overline{\mathcal{S}}^i(h; t_k))\,\mathrm{d}\theta. \end{aligned} \tag{2.9}$$

Therefore the inequality (2.8) can be written in the form

$$\left| \overline{s}_i(h; t) - \overline{s}_i(h; t_k) - \int_{t_k}^{t} f_i(\theta, \overline{\mathcal{S}}^i(h; t_k))\,\mathrm{d}\theta \right| \le \frac{1}{2} h^2 \Lambda_{i*} m. \tag{2.10}$$

Using (2.10), (2.2) and (2.7) we obtain

$$\begin{aligned} &\left| \overline{s}_i(h; t) - \overline{s}_i(h; t_k) - \int_{t_k}^{t} \overline{f}_i(\theta, \overline{\mathcal{S}}^i(h; \theta))\,\mathrm{d}\theta \right| \\ &\quad \le \left| \overline{s}_i(h; t) - \overline{s}_i(h; t_k) - \int_{t_k}^{t} \overline{f}_i(\theta, \overline{\mathcal{S}}^i(h; t_k))\,\mathrm{d}\theta \right| \\ &\qquad + \left| \int_{t_k}^{t} \Big(\overline{f}_i(\theta, \overline{\mathcal{S}}^i(h; t_k)) - \overline{f}_i(\overline{\mathcal{S}}^i(h; \theta)) \Big)\,\mathrm{d}\theta \right| \\ &\quad \le \frac{1}{2} h^2 \Lambda_{i*} m + \int_{t_k}^{t} \Lambda_{i*} |\overline{\mathcal{S}}^i(h; t_k) - \overline{\mathcal{S}}^i(h; \theta)|\,\mathrm{d}\theta \\ &\quad \le \frac{1}{2} h^2 \Lambda_{i*} m + \frac{1}{2} h^2 \Lambda_{i*} m = h^2 \Lambda_{i*} m. \end{aligned}$$

Let now $t \in [t_0, \bar{t}]$. There exist an interval $[t_r, t_{r+1}]$ such that $t \in [t_r, t_{r+1}]$, $r \in \{0, 1, \ldots, p-1\}$. Applying the above inequality for the intervals $[t_0, t_1], [t_1, t_2], \ldots$, $[t_r, t]$ we have

$$\begin{aligned}
&\left|\bar{s}_i(h;t) - \bar{s}_i(h;t_0) - \int_{t_0}^{t} \bar{f}_i(\theta, \overline{\mathcal{S}}^i(h;\theta))\,\mathrm{d}\theta\right| \\
&\quad \le \sum_{k=0}^{r-1} \left|\bar{s}_i(h;t_{k+1}) - \bar{s}_i(h;t_k) - \int_{t_k}^{t_{k+1}} \bar{f}_i(\theta, \overline{\mathcal{S}}^i(h;\theta))\,\mathrm{d}\theta\right| \\
&\quad + \left|\bar{s}_i(h;t) - \bar{s}_i(h;t_r) - \int_{t_r}^{t} \bar{f}_i(\theta, \overline{\mathcal{S}}^i(h;\theta))\,\mathrm{d}\theta\right| \le p\,h^2 \Lambda_{i*} m,
\end{aligned}$$

which yields

$$\left|\bar{s}_i(h;t) - \bar{s}_i(h;t_0) - \int_{t_0}^{t} \bar{f}_i(\theta, \overline{\mathcal{S}}^i(h;\theta))\,\mathrm{d}\theta\right| \le h(\bar{t} - t_0)\Lambda_{i*} m, \quad i = 1, \ldots, n. \tag{2.11}$$

In a similar way we obtain

$$\left|\underline{s}_i(h;t) - \underline{s}_i(h;t_0) - \int_{t_0}^{t} \underline{f}_i(\theta, \underline{\mathcal{S}}^i(h;\theta))\,\mathrm{d}\theta\right| \le h(\bar{t} - t_0)\Lambda_{i*} m, \quad i = 1, \ldots, n. \tag{2.12}$$

It is easy to see that the functions in each of the sets $\{\underline{s}(h;\cdot)\}$ and $\{\bar{s}(h;\cdot)\}$ are uniformly bounded and equicontinuous. Then the theorem of Arzelá-Ascoli implies that $\{\underline{s}(h;\cdot)\}$ and $\{\bar{s}(h;\cdot)\}$ considered as generalized sequences of h, $h \to 0$, have subsequences $\{\underline{s}(h_\alpha;\cdot)\}$ and $\{\bar{s}(h_\alpha;\cdot)\}$ that are uniformly convergent to continuous functions $\underline{s}$ and $\bar{s}$ respectively. Obviously $\underline{s} \le \bar{s}$. Let $\mathcal{S} = [\underline{s}, \bar{s}]$. From (2.11) and (2.12) when $h = h_\alpha \to 0$ it follows that

$$\begin{aligned}
\underline{s}_i(t) &= \underline{s}_i(t_0) + \int_{t_0}^{t} \underline{f}_i(\theta, \underline{\mathcal{S}}^i(\theta))\,\mathrm{d}\theta, \quad i = 1, \ldots, n, \\
\bar{s}_i(t) &= \bar{s}_i(t_0) + \int_{t_0}^{t} \bar{f}_i(\theta, \overline{\mathcal{S}}^i(\theta))\,\mathrm{d}\theta, \quad i = 1, \ldots, n,
\end{aligned}$$

which implies that $\mathcal{S}$ is differentiable and

$$\begin{aligned}
\mathcal{L}\mathcal{S}(t) &= 0, \qquad t \in [t_0, \bar{t}], \\
\mathcal{S}(t_0) &= X^0.
\end{aligned}$$

Therefore $\mathcal{S}(t) = X(t_0, X^0; t)$, $t \in [t_0, \bar{t}]$.

Since this is true for any other convergent subsequences of $\{\underline{s}(h;\cdot)\}$ and $\{\bar{s}(h;\cdot)\}$ then $\underline{x}(t_0, X^0;\cdot)$ is the only accumulation point of $\{\underline{s}(h;\cdot)\}$ and $\bar{x}(t_0, X^0;\cdot)$ is the only accumulation point of $\{\bar{s}(h;\cdot)\}$. Therefore

$$\lim_{h \to 0} \mathcal{S}(h;t) = X(t_0, X^0; t). \tag{2.13}$$

This concludes the proof because both statements of the theorem follow from (2.5) and (2.13). □

THEOREM 2.2. *If a numerical method produces interval enclosures $S(h;t)$ of the solution of problem (1.1)–(1.2) such that $S(h;t)$ satisfies the wrapping property at the points of the mesh $\{t_0, t_1, \dots, t_n\}$ and the local error is*

$$\left|S(h;t) -^{-} \left[x\left(t_k, S(h;t_k);t\right)\right]\right| = o(h), \quad t \in [t_k, t_{k+1}], \quad k = 0, 1, \dots, n-1,$$

then

$$\lim_{h\to 0} S(h;t) = \widehat{X}(t), \quad t \in [t_0, \bar{t}].$$

Proof. Using standard techniques one can show that the limit of $S(h;t)$ is the same as the limit of $\boldsymbol{S}(h,t)$ and then the statement follows from Theorem 2.1. □

Theorem 2.2 shows that in general the interval enclosures produced by a method of the considered type do not converge to the optimal interval enclosure $[x(t_0, X^0;t)]$ of the solution but to the wrapping function $\widehat{X}(t)$. Convergence to $[x(t_0, X^0;t)]$ is obtained if and only if $[x(t_0, X^0;t)] = \widehat{X}(t)$. More precise analysis can reveal that when $[x(t_0, X^0;t)] \neq \widehat{X}(t)$ the rate of convergence is $O(h)$ irrespective of the rate of the local approximation while if $[x(t_0, X^0;t)] = \widehat{X}(t)$ the rate of convergence corresponds to the rate of local approximation.

3. Applications

Using the concept of wrapping function we can quantify the wrapping effect associated with problem (1.1)–(1.2) in the following way. Let $S(h;t)$ be an interval enclosure of the solution of (1.1)–(1.2) produced by a method of a propagate and wrap type. The limit of the error of approximation when $h \to 0$ is

$$\lim_{h\to 0} \wp\left(S(h;t), [x(t_0, X^0;t)]\right) = \wp\left(\widehat{X}(t), [x(t_0, X^0;t)]\right).$$

The quantity

$$\wp\left(\widehat{X}(t), [x(t_0, X^0;t)]\right) = \| \, |\widehat{X}(t) -^{-} [x(t_0, X^0;t)]| \, \| \tag{3.1}$$

does not depend on the method and characterizes problem (1.1)–(1.2) with respect to the occurrence of the wrapping effect and its magnitude. Therefore it can be used as a measure of the wrapping effect associated with problem (1.1)–(1.2). The vector function

$$|\widehat{X}(t) -^{-} [x(t_0, X^0;t)]| \tag{3.2}$$

provides more detailed information about the wrapping effect because its coordinates give the magnitude of the wrapping effect in the corresponding coordinate directions

$$|\widehat{X}_i(t) -^{-} [x_i(t_0, X^0;t)]| = \wp\left(\widehat{X}_i(t), [x_i(t_0, X^0;t)]\right), \quad i = 1, \dots, n.$$

Each one of the functions (3.1) and (3.2) may be used in characterizing the wrapping effect associated with a particular problem.

If $\widehat{X}(t) = [x(t_0, X^0; t)]$ for a problem of the form (1.1)–(1.2) we say that this problem *has no wrapping effect* (or: *has zero wrapping effect*, or: *is without wrapping effect*) because the enclosures produced by any method of propagate and wrap type converge to the optimal interval enclosure with a rate corresponding to the rate of local approximation provided by the method. For a problem without wrapping effect the functions (3.1) and (3.2) are equal to zero.

REVISITING EXAMPLE 1

The exact solution of the system of linear equations (1.3) is

$$x(0, x^0; t) = \begin{pmatrix} e^{-2t} & 0 & 0 \\ 2(e^{-t} - e^{-2t}) & \cosh t & -\sinh t \\ 2(e^{-t} - e^{-2t}) & -\sinh t & \cosh t \end{pmatrix} x^0.$$

Therefore for the optimal interval enclosure $[x(0, X^0; t)] = [\underline{x}(0, X^0; t), \overline{x}(0, X^0; t)]$ we have

$$\underline{x}(t_0, X^0; t) = \begin{pmatrix} \underline{x}_1^0 e^{-2t} \\ 2\underline{x}_1^0(e^{-t} - e^{-2t}) + \underline{x}_2^0 \cosh t - \overline{x}_3^0 \sinh t \\ 2\underline{x}_1^0(e^{-t} - e^{-2t}) - \overline{x}_2^0 \sinh t + \underline{x}_3^0 \cosh t \end{pmatrix} \tag{3.3}$$

and

$$\overline{x}(t_0, X^0; t) = \begin{pmatrix} \overline{x}_1^0 e^{-2t} \\ 2\overline{x}_1^0(e^{-t} - e^{-2t}) + \overline{x}_2^0 \cosh t - \underline{x}_3^0 \sinh t \\ 2\overline{x}_1^0(e^{-t} - e^{-2t}) - \underline{x}_2^0 \sinh t + \overline{x}_3^0 \cosh t \end{pmatrix}. \tag{3.4}$$

The right-hand side of the equation in Example 1 has the following interval extension

$$f(t, [\underline{x}, \overline{x}]) = \begin{pmatrix} [-2\overline{x}_1, -2\underline{x}_1] \\ [2\underline{x}_1 - \overline{x}_3, 2\overline{x}_1 - \underline{x}_3] \\ [2\underline{x}_1 - \overline{x}_2, 2\overline{x}_1 - \underline{x}_2] \end{pmatrix}.$$

Therefore Problem (2.3)–(2.4) can be written in the form

$$\begin{aligned} \dot{\underline{x}}_1 &= -\underline{x}_1, & \dot{\overline{x}}_1 &= -2\overline{x}_1, \\ \dot{\underline{x}}_2 &= 2\underline{x}_1 - \overline{x}_3, & \dot{\overline{x}}_2 &= 2\overline{x}_1 - \underline{x}_3, \\ \dot{\underline{x}}_3 &= 2\underline{x}_1 - \overline{x}_2, & \dot{\overline{x}}_3 &= 2\overline{x}_1 - \underline{x}_2, \\ \underline{x}_i(0) &= 1 - \varepsilon_i, \quad i = 1, 2, 3, & \overline{x}_i(0) &= 1 + \varepsilon_i, \quad i = 1, 2, 3. \end{aligned}$$

The above problem can be solved using standard techniques and its solution gives the wrapping function $\widehat{X} = [\underline{\widehat{x}}, \overline{\widehat{x}}]$, where $\underline{\widehat{x}}(t)$ is the vector

$$\begin{pmatrix} \underline{x}_1^0 e^{-2t} \\ \frac{1}{3}\underline{x}_1^0(e^t+3e^{-t}-4e^{-2t}) - \frac{1}{3}\overline{x}_1^0(e^t-3e^{-t}+2e^{-2t}) + \underline{x}_2^0\cosh t - \overline{x}_3^0\sinh t \\ \frac{1}{3}\underline{x}_1^0(e^t+3e^{-t}-4e^{-2t}) - \frac{1}{3}\overline{x}_1^0(e^t-3e^{-t}+2e^{-2t}) - \overline{x}_2^0\sinh t + \underline{x}_3^0\cosh t \end{pmatrix} \quad (3.5)$$

and $\overline{\widehat{x}}(t)$ is

$$\begin{pmatrix} \overline{x}_1^0 e^{-2t} \\ \frac{1}{3}\overline{x}_1^0(e^t+3e^{-t}-4e^{-2t}) - \frac{1}{3}\underline{x}_1^0(e^t-3e^{-t}+2e^{-2t}) + \overline{x}_2^0\cosh t - \underline{x}_3^0\sinh t \\ \frac{1}{3}\overline{x}_1^0(e^t+3e^{-t}-4e^{-2t}) - \frac{1}{3}\underline{x}_1^0(e^t-3e^{-t}+2e^{-2t}) - \underline{x}_2^0\sinh t + \overline{x}_3^0\cosh t \end{pmatrix}. \quad (3.6)$$

Using (3.3), (3.4), (3.5), and (3.6) we obtain the measure of the wrapping effect (3.2)

$$\begin{aligned} \left|\widehat{X}(t) -^{-} [x(0, X^0; t)]\right| &= \begin{pmatrix} 0 \\ \frac{1}{3}(e^t - 3e^{-t} + 2e^{-2t})(\overline{x}_1^0 - \underline{x}_1^0) \\ \frac{1}{3}(e^t - 3e^{-t} + 2e^{-2t})(\overline{x}_1^0 - \underline{x}_1^0) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \frac{2}{3}(e^t - 3e^{-t} + 2e^{-2t})\varepsilon_1 \\ \frac{2}{3}(e^t - 3e^{-t} + 2e^{-2t})\varepsilon_1 \end{pmatrix}. \end{aligned}$$

From the above form of the wrapping effect measure we can make the following observations:

1. There is no wrapping effect in x_1 (see Figure 1). This is not surprising because x_1 is obtained only from the first equation and the right hand side of a single equation is always quasi-isotone.
2. The wrapping effect in x_2 and x_3 depends only on the width of X_1^0. Therefore there is no wrapping effect if $w(X_1^0) = 0$ (see Figure 2).

On Figure 3, where the computed enclosures for Problem (1.3)–(1.4) are plotted together with the wrapping function, convergence of the enclosures to the wrapping function can be observed.

MOORE'S EXAMPLE

The following example is often discussed in literature.

$$\begin{aligned} x_1 &= x_2, & x_1(0) &= x_1^0 \in X_1^0 = [-\delta, \delta], \\ x_2 &= -x_1, & x_2(0) &= x_2^0 \in X_2^0 = [1 - \delta,\ 1 + \delta]. \end{aligned} \quad (3.7)$$

R. Moore showed [7] that at $t = 2\pi$ the computed interval enclosures are inflated by a factor of approximately $e^{2\pi} \approx 535$. We shall obtain this result using the wrapping function of problem (3.7).

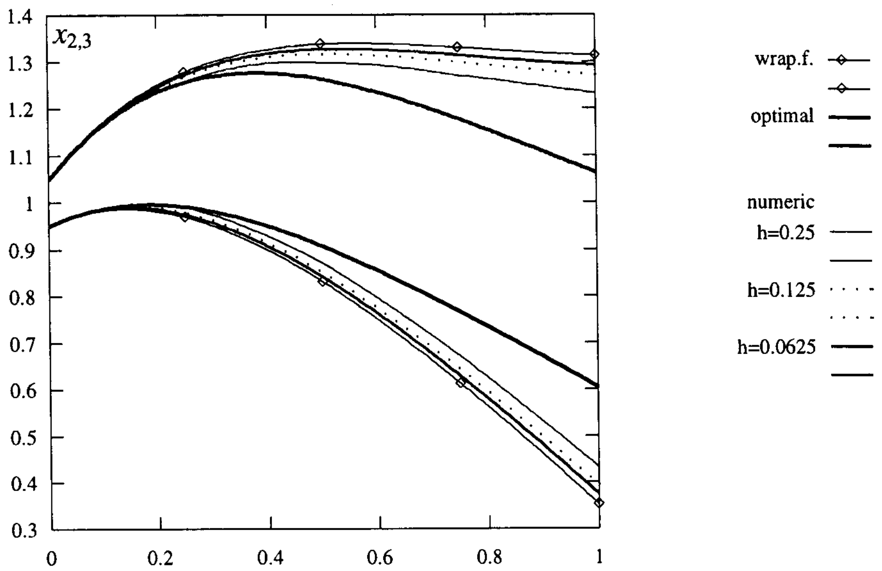

Figure 3. *Problem (1.3) with* $\varepsilon_1 = 0.2$, $\varepsilon_2 = \varepsilon_3 = 0.05$. *Wrapping function, optimal enclosure and enclosures computed numerically for various step sizes h.*

The exact solution of this problem is

$$x(0, x^0; t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} x^0.$$

Hence the optimal interval enclosure can be represented in the form

$$[x(0; x^0; t)] = \begin{pmatrix} (\cos t)X_1^0 + (\sin t)X_2^0 \\ -(\sin t)X_1^0 + (\cos t)X_2^0 \end{pmatrix}$$

and its width is

$$\begin{aligned} w([x(0; x^0; t)]) &= \begin{pmatrix} |\cos t| w(X_1^0) + |\sin t| w(X_2^0) \\ |\sin t| w(X_1^0) + |\cos t| w(X_2^0) \end{pmatrix} \\ &= (|\cos t| + |\sin t|) \begin{pmatrix} 2\delta \\ 2\delta \end{pmatrix}. \end{aligned} \tag{3.8}$$

Problem (2.3)–(2.4) for system (3.7) can be written in the following form

$$\begin{aligned} \dot{\underline{x}}_1 &= \underline{x}_2, & \underline{x}_1(0) &= -\delta, \\ \dot{\overline{x}}_1 &= \overline{x}_2, & \overline{x}_1(0) &= \delta, \\ \dot{\underline{x}}_2 &= -\overline{x}_1, & \underline{x}_2(0) &= 1 - \delta, \\ \dot{\overline{x}}_2 &= -\underline{x}_1, & \overline{x}_2(0) &= 1 + \delta. \end{aligned}$$

Solving this problem we obtain the wrapping function $\widehat{X}(t) = [\widehat{\underline{x}}(t), \widehat{\overline{x}}(t)]$ of (3.7); for $2\widehat{\underline{x}}(t)$ we have

$$\begin{pmatrix} (\cos t{+}\cosh t)\underline{x}_1^0 + (\cos t{-}\cosh t)\overline{x}_1^0 + (\sin t{+}\sinh t)\underline{x}_2^0 + (\sin t{-}\sinh t)\overline{x}_2^0 \\ (-\sin t{+}\sinh t)\underline{x}_1^0 - (\sin t{+}\sinh t)\overline{x}_1^0 + (\cos t{+}\cosh t)\underline{x}_2^0 + (\cos t{-}\cosh t)\overline{x}_2^0 \end{pmatrix}$$

and for $2\widehat{\overline{x}}(t)$

$$\begin{pmatrix} (\cos t{-}\cosh t)\underline{x}_1^0 + (\cos t{+}\cosh t)\overline{x}_1^0 + (\sin t{-}\sinh t)\underline{x}_2^0 + (\sin t{+}\sinh t)\overline{x}_2^0 \\ -(\sin t{+}\sinh t)\underline{x}_1^0 + (-\sin t{+}\sinh t)\overline{x}_1^0 + (\cos t{-}\cosh t)\underline{x}_2^0 + (\cos t{+}\cosh t)\overline{x}_2^0 \end{pmatrix}.$$

Therefore

$$\begin{aligned} w(\widehat{X}(t)) &= \widehat{\overline{x}}(t) - \widehat{\underline{x}}(t) \\ &= \begin{pmatrix} \cosh t(\overline{x}_1^0 - \underline{x}_1^0) + \sinh t(\overline{x}_2^0 - \underline{x}_2^0) \\ \sinh t(\overline{x}_1^0 - \underline{x}_1^0) + \cosh t(\overline{x}_2^0 - \underline{x}_2^0) \end{pmatrix} \\ &= e^t \begin{pmatrix} 2\delta \\ 2\delta \end{pmatrix}. \end{aligned} \tag{3.9}$$

From (3.8) and (3.9) we have

$$w(\widehat{X}(t)) = \frac{e^t}{|\cos t| + |\sin t|} w([x(t_0, x^0; t)]).$$

Since $\widehat{X}$ is the limit of the interval enclosures when $h \to 0$ then these enclosures are inflated at the point t approximately by a factor of $e^t / (|\cos t| + |\sin t|)$ when h is small enough. At $t = 2\pi$ the value of this factor is $e^{2\pi}$.

4. Problems without Wrapping Effect

It is clear from the previous sections that methods of propagate and wrap type can be applied successfully only to problems where no wrapping effect occurs. In this section we use the concept of wrapping function to characterize such problems. Our approach is to find problems of the form (1.1)–(1.2) such that the wrapping function $\widehat{X}(t)$ equals the optimal interval enclosure $[x(t_0, X^0; t)]$ or equivalently $w(\widehat{X}(t)) = w([x(t_0, X^0; t)])$, $t \in [t_0, \bar{t}]$.

THEOREM 4.1. *If there exists a diagonal matrix $Q = diag(q_1, q_2, \dots, q_n)$, $q_i \in \{-1, 1\}$, $i = 1, \dots, n$, such that the function $Qf(t, Qx)$ is a quasi-isotone function on $x \in QD = \{Qd : d \in D\}$ then the wrapping function $\widehat{X}$ of problem (1.1)–(1.2) equals the optimal interval enclosure $[x(t_0, X^0; \cdot)]$ i.e. there is no wrapping effect.*

Proof. Let us note that the linear transformation $\mathcal{Q} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $\mathcal{Q}(x) = Qx$, where $Q = diag(q_1, q_2, \dots, q_n)$, $q_i \in \{-1, 1\}$, preserves the intervals i.e. if $X \in I(\mathbb{R})^n$ then $QX = \{Qx : x \in X\} \in I(\mathbb{R})^n$ or in general if $X \subset \mathbb{R}^n$ then

$$[QX] = Q[X]. \tag{4.1}$$

where $[X]$ is the optimal (tightest) interval containing the set X and $[QX]$ is the optimal (tightest) interval containing the set QX.

We consider the problem

$$\begin{aligned} \dot{y} &= g(t,y), & (4.2)\\ y(t_0) &= y_0 \in Y^0 = QX^0, & (4.3) \end{aligned}$$

where $g(y) = Qf(t,Qy)$ is a quasi-isotone function of $y \in QD$. For the wrapping function $\widehat{Y} = [\widehat{\underline{y}}, \widehat{\overline{y}}]$ of this problem we have for $i = 1, \dots, n$

$$\begin{aligned} \dot{\widehat{\underline{y}}}_i &= \underline{g}_i(t, [\widehat{\underline{y}}_1, \widehat{\overline{y}}_1], \dots, [\widehat{\underline{y}}_{i-1}, \widehat{\overline{y}}_{i-1}], \widehat{\underline{y}}_i, [\widehat{\underline{y}}_{i+1}, \widehat{\overline{y}}_{i+1}], \dots, [\widehat{\underline{y}}_n, \widehat{\overline{y}}_n]),\\ \dot{\widehat{\overline{y}}}_i &= \overline{g}_i(t, [\widehat{\underline{y}}_1, \widehat{\overline{y}}_1], \dots, [\widehat{\underline{y}}_{i-1}, \widehat{\overline{y}}_{i-1}], \widehat{\overline{y}}_i, [\widehat{\underline{y}}_{i+1}, \widehat{\overline{y}}_{i+1}], \dots, [\widehat{\underline{y}}_n, \widehat{\overline{y}}_n]), & (4.4)\\ \widehat{\underline{y}}(t_0) &= \underline{y}^0,\\ \widehat{\overline{y}}(t_0) &= \overline{y}^0. \end{aligned}$$

Since g_i is non-decreasing on $y_j, j \neq i$, we have for $i = 1, \dots, n$

$$\begin{aligned} &\underline{g}_i(t, [\widehat{\underline{y}}_1, \widehat{\overline{y}}_1], \dots, [\widehat{\underline{y}}_{i-1}, \widehat{\overline{y}}_{i-1}], \widehat{\underline{y}}_i, [\widehat{\underline{y}}_{i+1}, \widehat{\overline{y}}_{i+1}], \dots, [\widehat{\underline{y}}_n, \widehat{\overline{y}}_n])\\ &\qquad = g_i(t, \widehat{\underline{y}}_1, \dots, \widehat{\underline{y}}_{i-1}, \widehat{\underline{y}}_i, \widehat{\underline{y}}_{i+1}, \dots, \widehat{\underline{y}}_n),\\ &\overline{g}_i(t, [\widehat{\underline{y}}_1, \widehat{\overline{y}}_1], \dots, [\widehat{\underline{y}}_{i-1}, \widehat{\overline{y}}_{i-1}], \widehat{\overline{y}}_i, [\widehat{\underline{y}}_{i+1}, \widehat{\overline{y}}_{i+1}], \dots, [\widehat{\underline{y}}_n, \widehat{\overline{y}}_n])\\ &\qquad = g_i(t, \widehat{\overline{y}}_1, \dots, \widehat{\overline{y}}_{i-1}, \widehat{\overline{y}}_i, \widehat{\overline{y}}_{i+1}, \dots, \widehat{\overline{y}}_n). \end{aligned}$$

Then from (4.4) it follows that $\widehat{\underline{y}}(t) = y(t_0, \underline{y}^0; t)$ and $\widehat{\overline{y}}(t) = y(t_0, \overline{y}^0; t)$ belong to the solution $y(t_0, Y^0; t)$ of Problem (4.2)–(4.3) which implies that

$$\widehat{Y}(t) = [y(t_0, Y^0; t)] = [y(t_0, QX^0; t)], \quad t \in [t_0, \bar{t}].$$

For every solution $x(t_0, x^0; t)$ of equation (1.1) we have

$$x(t_0, x^0; t) = Qy(t_0, Qx^0; t).$$

Therefore from (4.1) it follows that

$$[x(t_0, X^0; t)] = [Qy(t_0, QX^0; t)] = Q[y(t_0, QX^0; t)] = Q\widehat{Y}(t). \tag{4.5}$$

It remains to prove that $Q\widehat{Y}$ is the wrapping function of Problem (1.1)–(1.2).

At $t = t_0$ we have $Q\widehat{Y}(t_0) = Q^2X^0 = X^0$. Let $\theta \in [t_0, \bar{t})$ and let $u \in Q\widehat{Y}(\theta)$. Then $Qu \in \widehat{Y}(\theta)$ and

$$x(\theta, u; t) = Qy(\theta, Qu; t) \in Q\widehat{Y}(t), \quad t \in [\theta, \bar{t}],$$

i.e. $Q\widehat{Y}$ satisfies the wrapping property at every $\theta \in [t_0, \bar{t})$. Therefore

$$[x(t_0, X^0; t)] \subset \widehat{X}(t) \subset Q\widehat{Y}(t), \quad t \in [t_0, \bar{t}].$$

Then (4.5) implies

$$[x(t_0, X^0; t)] = \widehat{X}(t) = Q\widehat{Y}(t), \quad t \in [t_0, \bar{t}],$$

which concludes the proof of the theorem. □

Considering again Example 1 and Moore's example we can see that Theorem 4.1 is not applicable to either of them because a matrix Q with the required properties does not exist. Theorem 4.1 is applicable to the following example.

EXAMPLE 2. Consider the problem

$$\begin{aligned} \dot{x}_1 &= -2x_1, & x_1(0) &= x_1^0 \in X_1^0 = 1 + [-\varepsilon_1, \varepsilon_1], \\ \dot{x}_2 &= 2x_1 - x_3, & x_2(0) &= x_2^0 \in X_2^0 = 1 + [-\varepsilon_2, \varepsilon_2], \\ \dot{x}_3 &= -2x_1 - x_2, & x_3(0) &= x_3^0 \in X_3^0 = 1 + [-\varepsilon_3, \varepsilon_3], \end{aligned} \tag{4.6}$$

in the interval [0, 1]. The function

$$f(x) = \begin{pmatrix} -2x_1 \\ 2x_1 - x_3 \\ -2x_1 - x_2 \end{pmatrix}$$

can be transformed into a quasi-isotone function using a matrix

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Indeed,

$$Qf(t, Qx) = \begin{pmatrix} -2x_1 \\ 2x_1 + x_3 \\ 2x_1 + x_2 \end{pmatrix}$$

is quasi-isotone. Then from Theorem 4.1 it follows that problem (4.6) is a problem without wrapping effect for any initial condition $X^0 \in I(\mathbb{R})^n$. This theoretical result is consistent with our numerical experiments. Consider the values of ε_i, $i = 1, 2, 3$, given by (1.4) and apply the same method as in Example 1. While in the case of Example 1 we obtain enclosures which diverge from the optimal enclosure (Figure 1, right) the enclosures produced by the method in the case of Example 2 converge to the optimal one. This is demonstrated graphically on Figure 4. Since the optimal and the numerically computed enclosures for x_1 are the same as in Example 1 the corresponding graphs are omitted; only the graphs for x_2 and x_3 are presented. The graphs of the computed enclosures are visually undistinguishable from the optimal enclosure. At the bottom of Figure 4 the error functions

$$\wp\left(S_i(h; t), [x_i(t_0, X^0; t)]\right), \quad i = 1, 2, 3$$

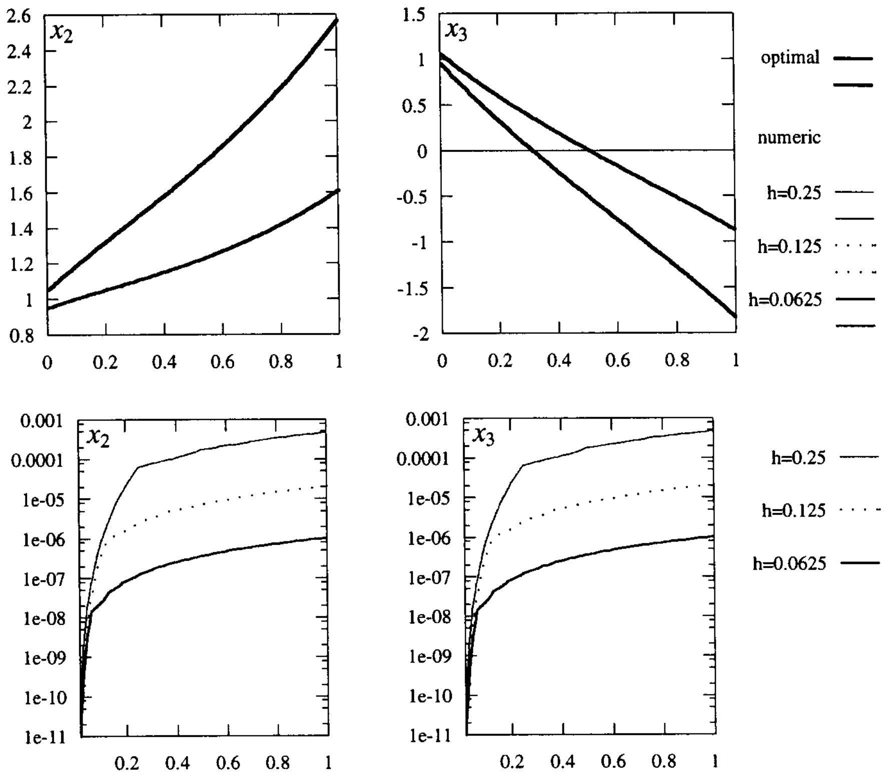

Figure 4. *Problem (4.6) with $\varepsilon_1 = 0.2$, $\varepsilon_2 = \varepsilon_3 = 0.05$. Optimal enclosure and enclosures computed numerically for various step sizes h (top) and errors of the computed enclosures on a logarithmic scale (bottom).*

are plotted on a logarithmic scale. Convergence at a rate consistent with the expected rate of global convergence can be observed.

Let us note that Theorem 4.1 provides a sufficient condition for problems with no wrapping effect. It is an interesting question if and in what form this condition is also a necessary condition for having no wrapping effect. Results in this regard will be discussed in a separate paper.

5. Conclusion

In this paper a new concept of wrapping function is defined using the notion of enclosing (wrapping) sets in $\mathbb{R}^n$ by n-dimentional intervals (of course other classes of sets in $\mathbb{R}^n$ like balls or ellipsoids can be used as well). It is proved that the wrapping function is a solution of a differential problem of the form (2.3)–(2.4). The wrapping function is the limit of the interval enclosures for the solution of Problem (1.1)–(1.2) produced by methods of propagate and wrap type (Theorem 2.2). Therefore the wrapping function can be used in quantitative estimation of the wrapping effect for initial value problems of ODEs; some applications in this regard are discussed by means of examples. The wrapping function is also a useful tool in characterizing problems without wrapping effect as demonstrated in Theorem 4.1.

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