Reminiscences
"Back in the Good Old Days..."
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The Mystery of Intervals

by Svetoslav Markov

Interval functions have often been discussed in relation to Hausdorff approximations [6] at Blagovest Sendov's seminar on Approximation Theory held regularly in the Bulgarian Academy of Sciences since 1964. Numerical computations related to the best polynomial Hausdorff approximations of certain interval functions require special attention to round-off errors [2]. In 1975 Sendov, who was my PhD supervisor, gave me reprints of papers by T. Sunaga, H. Ratschek and G. Schröder on interval arithmetic and differentiation of interval functions. I was very impressed by these papers, especially by the one from Sunaga, which I enjoyed studying thoroughly [5].

It took many centuries to human mind to grasp the mystery of numbers. G. Birkhoff notes: "*We should not forget that zero and negative numbers were among the last to be accepted*" [1]. The primary use of negative numbers (and zero) is to make the equation $A + X = B$ always solvable, i.e. to make the additive semigroup $(\mathbb{R}^+, +)$ of nonnegative numbers a group. The isomorphic extension (embedding) of an Abelian semigroup into a group is now a common mathematical tool.

The set $\mathbb{I}_+ = I(\mathbb{R})$ of (proper) intervals with addition (multiplication) is an Abelian semigroup, and thus can be made (embedded in, extended to) a group; this has been noticed by M. Warmus in 1956 [7]. The new elements involved are the *improper* intervals making a set $\mathbb{I}_-$, which together with the set $\mathbb{I}_+$ forms the set $\mathbb{I}$ of all (proper and improper) intervals. An isomorphic extension of multiplication (addition is trivial) from $\mathbb{I}_+$ to $\mathbb{I}$ based on set-theoretic arguments (i.e. preserving inclusion isotonicity) is proposed by H.-J. Ortolf in 1969. In the beginning of the seventies O. Mayer, H. Ratschek, G. Schröder and M. Kracht studied abstractly the algebraic structure of intervals with respect to addition and multiplication by scalar, known as *quasilinear space*. A rigorous algebraic study of the interval arithmetic in $\mathbb{I}$ is due to E. Kaucher [3]; followed by related work by J. Wolff von Gudenberg, E. Gardeñes et al.

My early work is related to inner operations in $\mathbb{I}_+$. To give a brief idea of these operations it is instructive to compare the algebraic properties of proper intervals to those of nonnegative numbers. Recall the useful operation " " in the additive semigroup $(\mathbb{R}^+, +)$ of nonnegative numbers: $A \quad B$ is the solution X of $A + X = B$, if X exists, and is the solution Y of $B + Y = A$, if Y exists. Of course $A \quad B = |A - B|$, but we do not have negative numbers $-B$ in $\mathbb{R}^+$, and hence we

cannot write $A - B = A + (-B)$. The so-called *inner (nonstandard)* operations for proper intervals are defined similarly. For example, the interval *inner difference* $A \quad B, A, B \in \mathbb{I}_+$, is the solution X of $A + X = B$, if such a solution X exists, and is $-Y = (-1)Y$, where Y is the solution of $B + Y = A$, provided Y exists; if both solutions exist, they coincide. Why do we take $-Y$ in the above definition, and not Y, as in the case of numbers? The answer of this question hides some of the secrets of intervals.

By the end of the eighties many properties and applications of inner operations were established—mainly for ranges of monotone functions and consequently for a number of numerical algorithms. In my work I was supported by N. Dimitrova, E. Popova, R. Anguelov, P. Bochev, G. Grozev, and other collaborators. Meanwhile, I was searching for a link between the (inner) operations in $\mathbb{I}_+$ and the operations in $\mathbb{I}$. The success came with the introduction of a suitable symbolic notation involving the functions $\sigma : \mathbb{R} \longrightarrow \{+, -\}$ defined by $\sigma(\alpha) = \{+, \text{ if } \alpha \geq 0; \ -, \text{ if } \alpha < 0\}$ and $\tau : \mathbb{I} \longrightarrow \{+, -\}$ defined by $\tau(a) = \{+, \text{ if } a \text{ is proper}; \ -, \text{ if } a \text{ is improper}\}$. The functions σ (sign of a real number) and τ (type, direction of an interval) often appear in interval arithmetic. Using σ we obtain a concise, uniform and functional notation for the inner interval operations. For instance, we write $A -^\sigma B$, where $A -^+ B = A - B, A -^- B = A \quad B$ and $A +^\sigma B = A -^\sigma (-B)$. By means of σ the distributive law for proper intervals (when using multiplication by scalar) takes the form $(\alpha + \beta) * C = \alpha * C +^{\sigma(\alpha\beta)} \beta * C$. Now, the desired link: for $a, b \in \mathbb{I}$ we have $\text{prop}(a + b) = \text{prop}(a) +^{\tau(a)\tau(b)} \text{prop}(b)$, where $\text{prop}(a) \in \mathbb{I}_+$ is the proper interval corresponding to $a \in \mathbb{I}$; similar relations exist for other operations.

Like numbers, intervals can be added, multiplied and compared, but differently to numbers, which have the order relation "$\leq$", intervals have an additional order *inclusion* $\subseteq$. We can summarize the properties of numbers by saying that they make an ordered field, but we still do not know what precisely is the ordered "field-like" system of intervals $(\mathbb{I}, +, *, \leq, \subseteq)$. Like numbers intervals from $\mathbb{I}$ have group properties with respect to both addition and multiplication. In the Abelian group $(\mathbb{I}, +)$ every $c \in \mathbb{I}$ has an opposite $-c$. For $c \in \mathbb{I}$ write $c_+ = c$ and call $c_- = (-1)(-c)$ the *conjugate (dual)* interval to c; note that in general $(-1)(-c) = c$; another interval secret! For any $c \in \mathbb{I}$, the corresponding proper interval is $\text{prop}(c) = c_{\tau(c)}$.

Some secrets related to the distributivity-like relations have been recently revealed. In the additive group of intervals with multiplication by scalar $(\mathbb{I}, +, \mathbb{R}, *)$ there is a simple distributivity relation: $(\alpha + \beta) * c_{\sigma(\alpha+\beta)} = \alpha * c_{\sigma(\alpha)} + \beta * c_{\sigma(\beta)}$, which enables a full characterization of this so-called *quasilinear space with group structure*. Namely, one can define linear dependence, basis, etc. in the same manner as done in a linear space, and one can show that a quasilinear space with group structure is a direct sum of a linear and a symmetric space [4]. Symmetric intervals in $\mathbb{I}$ have the property $c_- = -c$ and obey the distributive relation $(a + b) * c_{\tau(a+b)} = a * c_{\tau(a)} + b * c_{\tau(b)}$. This relation is a special case of more general distributive-like relations obtained by E. Popova; it can be used for an abstract definition of an

ordered field-like system, which may become a key structure for further understanding the mystery of intervals.

References

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