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On the Algebraic Properties of Errors

In numerical analysis (absolute) errors can be identified with one-dimensional intervals symmetric with respect to zero. Addition, multiplication and inclusion of errors are well-defined (set-theoretically) in interval analysis. We study axiomatically the algebraic properties of such a system of errors. To this end we introduce and investigate a new abstract algebraic structure called e-ring.

1. Axiomatic Definition of an E-ring

Absolute errors are nonnegative real numbers $\delta > 0$, which can be identified with symmetric intervals of the form $[-\delta,\delta] \subseteq \mathbb{R}$. Any interval on $\mathbb{R}$ can be presented as the sum of a real number and an error, e. g. $[1,3] = 2 + [-1,1]$. Intervals, in particular errors, can be added, multiplied and ordered. However, the arithmetic of intervals, resp. errors, is still not completely known. The arithmetic of errors is different from the one for numbers as far as multiplication is concerned, e. g. multiplying an error by -1 does not influence the result: $-1 * [-\delta,\delta] = [-\delta,\delta]$. Like numbers, errors form an additive abelian monoid with cancellation law, hence can be embedded in a group; thereby new (improper, "negative") elements are introduced. One can extend inclusion and multiplication in such a way that the latter is isotone with respect to inclusion [1]. The system of errors involving addition, (inner) multiplication and the natural order (inclusion) can be axiomatically defined as follows.

Definition 1. An *e-ring (error ring)* is a system $(\mathbf{S},+,\circ,\subseteq)$, such that: A1. $(\mathbf{S},\subseteq)$ is a linearly ordered set; A2. $(\mathbf{S},+,\subseteq)$ is an isotone abelian group; A3. $(\mathbf{S},\circ,\subseteq)$ is an isotone abelian semigroup; A4. multiplication "$\circ$" in $(\mathbf{S},+,\circ,\subseteq)$ preserves the opposite operator and is distributive over a sum of two elements whenever the three elements involved either include the null (zero) of $(\mathbf{S},+)$ or are included in zero.

For comparison we recall that the familiar linearly ordered (l. o.) ring (field) of reals $\mathbb{R}_D = (\mathbb{R},+,\cdot,\leq)$ satisfies A1. and A2. Concerning assumption A3., $(\mathbb{R},\cdot,\leq)$ is an abelian semigroup, which is not isotone, as $a \leq b \Longrightarrow ac \leq bc$ does not hold for all a,b,c. The semigroup $(\mathbb{R},\cdot)$ is a monoid (has identity), which is not assumed to hold for the semigroup $(\mathbf{S},\circ)$. As far as A4. is concerned, the ring $(\mathbb{R},+,\cdot)$ is distributive over the sum of any two elements, but does not preserve the opposite operator, as, $-(ab) = (-a)(-b)$ does not hold in general (rather $-(ab) = (-a)b = a(-b)!$), whereas $\mathrm{opp}(a \circ b) = \mathrm{opp}(a) \circ\ \mathrm{opp}(b)$ is assumed in $\mathbf{S}$.

The null of $(\mathbf{S},+)$ is denoted by 0; the opposite element $\mathrm{opp}(a)$ to $a \in \mathbf{S}$ is denoted symbolically by a_- (in interval and convex analysis the symbol "$-$" is used to denote multiplication by the scalar -1, which is generally different from opposite). Using the above notations, according to Definition 1 $(\mathbf{S},+,\circ,\subseteq)$ is an e-ring, if for all $a,b,c \in \mathbf{S}$:

A1. $a \subseteq a$; $a \subseteq b,\ b \subseteq a \Longrightarrow a = b$; $a \subseteq b,\ b \subseteq c \Longrightarrow a \subseteq c$; either $a \subseteq b$ or $b \subseteq a$ holds true;
A2. $(a+b)+c = a+(b+c)$; $a+b = b+a$; $a+0 = a$; $a + a_- = 0$; $a \subseteq b \Longrightarrow a+c \subseteq b+c$;
A3. $(a \circ b) \circ c = a \circ (b \circ c)$; $a \subseteq b \Longrightarrow a \circ c \subseteq b \circ c$; $a \circ b = b \circ a$;
A4. $(a \circ b)_- = a_- \circ b_-$; $(a+b) \circ c = a \circ c + b \circ c$ if either $a,b,c \subseteq 0$, or $a,b,c \supseteq 0$.

Consider the following operation for $a,b \in \mathbb{R}_D$, called *e-multiplication*:

$$a \circ b = \begin{cases} |a|b = a|b|, & \text{if } \ a \geq 0, b \geq 0 \text{ or } \ a < 0, b < 0, \\ 0, & \text{if } \ a \geq 0, b < 0 \text{ or } \ a < 0, b \geq 0, \end{cases} \tag{1}$$

where $|a| = \{a, \text{if } a \geq 0; -a, \text{if } a < 0\}$. The l. o. field $\mathbb{R}_D$ endowed with the operation (1) will be called the *extended field of reals*. Thus there are two operations for multiplication in the extended real field. Note that in the case $a < 0$, $b < 0$ we have for the e-product: $a \circ b = -ab = -|ab| < 0$, whereas $ab > 0$.

The set $\mathbb{R}$ is closed with respect to (1). Hence we can consider the system $\mathbb{R}_E = (\mathbb{R},+,\circ,\leq)$, having the operation "$\circ$" instead of "$\cdot$" together with the same operation "$+$" and the same order relation "$\leq$" as in $\mathbb{R}_D$.

Proposition 1. *The system $\mathbb{R}_E = (\mathbb{R},+,\circ,\leq)$ involving multiplication (1) is an e-ring.*

P r o p o s i t i o n 2. *The system* $\mathbb{R}_E = (\mathbb{R}, +, \circ, \leq)$ *is isomorphic to the system of generalized symmetric intervals on the real line (as introduced in [1];* $\leq$ *corresponds to inclusion* $\subseteq$*).*

2. Properties of the E-ring $(\mathbf{S}, +, \circ, \subseteq)$

The following familiar relations take place in any isotone (additive) group $(\mathbf{S}, +, \subseteq)$: P1.1. $a + c = b + c \Longrightarrow a = b$; P1.2. $a \subset b \Longrightarrow a + c \subset b + c$; P1.3. $a + c \subseteq b + c \Longrightarrow a \subseteq b$; P1.4. $a + a_- = 0$; P1.5. $0_- = 0$; P1.6. $(a_-)_- = a$; P1.7. $(a + b)_- = a_- + b_-$; P1.8. $a \subseteq b,\ c \subseteq d \Longrightarrow a + c \subseteq b + d$. P1.9. If $a \subset 0$, then $0 \subset a_-$, and conversely, if $0 \subset a$, then $a_- \subset 0$.

Define $\tau : \mathbf{S} \longrightarrow \Lambda = \{+, -\}$ by $\tau(a) = \{+, \text{if } 0 \subseteq a;\ -, \text{if } a \subset 0\}$. The following properties hold in any isotone (multiplicative) semigroup $(\mathbf{S}, \circ, \subseteq)$: P2.1. $a \subseteq b,\ c \subseteq d \Longrightarrow a \circ c \subseteq b \circ d$; P2.2. $a \subseteq 0,\ b \subseteq 0 \Longrightarrow a \circ b \subseteq 0$, resp. $0 \subseteq a$, $0 \subseteq b \Longrightarrow 0 \subseteq a \circ b$; P2.3. $a \circ c = b \circ c, \tau(a) = \tau(b) = \tau(c) \Longrightarrow a = b$; P2.4. $a \circ c \subseteq b \circ c, \tau(a) = \tau(b) = \tau(c) \Longrightarrow a \subseteq b$.

The following consequences from axiom A4. hold: P3.1. $(a_- \circ b)_- = a \circ b_-$, and $a \circ b_- + a_- \circ b = 0$; P3.2. $0 \circ c = c \circ 0 = 0$; P3.3. $a \subseteq 0 \subseteq b \Longrightarrow a \circ b = 0$; P3.4. $a \circ a_- = 0$. In particular, $0 \circ 0 = 0 \circ 0_- = 0$.

Denote $a_+ = a$, so that the symbol a_λ makes sense for $\lambda \in \Lambda = \{+, -\}$. In every e-ring $\mathbf{S}$ the following *quasidistributive law* holds true:

P r o p o s i t i o n 3. *For any* $a, b, c \in \mathbf{S}$ *we have:* $(a + b) \circ c_{\tau(a+b)} = a \circ c_{\tau(a)} + b \circ c_{\tau(b)}$.

3. Relation between a l. o. ring and an e-ring

Formula (1) defines *e-multiplication* in a l. o. ring (field) by the familiar multiplication. Similarly, we can define familiar multiplication in an e-ring by means of τ:

$$a \cdot b \quad = \quad a_{\tau(b)} \circ b_{\tau(a)}. \tag{2}$$

P r o p o s i t i o n 4. *Every l. o. ring generates via (1) a unique (up to isomorphism) e-ring and vice versa, every e-ring induces via (2) a unique l. o. ring.*

Let us note that in an extended ring we can use mixed notations, e. g. formula (1) can be written

$$a \circ b = \begin{cases} (ab)_\tau, & \text{if } \tau(a) = \tau(b) = \tau, \\ 0, & \text{if } \tau(a) = -\tau(b). \end{cases} \tag{3}$$

Proposition 4 shows that results from a l. o. ring can be reformulated in the induced e-ring by means of (2) and vice versa, results from an e-ring can be reformulated in the induced l. o. ring by means of (1) or (3).

Detailed proofs of Propositions 1–4 can be find in [2]. It has been also shown in [2] how to relate an e-ring with a symmetric q-linear space, see [3]–[4], in order to obtain a structure, called an e-algebra, abstracting properties of arbitrary generalized intervals (not necessarily symmetric ones).

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4. References

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