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ON THE INCLUSION PROPERTIES OF INTERVAL MULTIPLICATION: A DIAGRAMMATIC STUDY *

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Abstract.

Certain algebraic properties of familiar (set-theoretic) interval multiplication are studied diagrammatically. The centred interval multiplication operations (the long known outward multiplication and the newly proposed inward one) are defined and studied diagrammatically in some detail, especially with respect to inclusion isotonicity.

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1 Introduction.

It is well-known that self-validating methods for linear systems demand a consideration of all errors in the coefficients of the system including round-off errors. A possible approach to treat linear systems with error control is to formulate the system as a linear interval system, that is to consider the system together with intervals corresponding to the errors in the respective coefficients of the system and to make use of interval arithmetic [15]. The latter is an efficient tool for the construction of self-validating methods due to the inclusion isotonicity of interval arithmetic operations. However, the application of interval arithmetic to a verified solution of linear interval equations is intricate, because interval arithmetic does not permit a representation by coordinates so that interval operations, respectively interval algebraic problems, cannot be generally reduced into real problems to be solved separately for each coordinate [12].

Usually, intervals are used in the endpoints form, where an interval is represented by its lowest and uppermost values. However, in many practical and

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theoretical situations it is more natural to use the midpoint-radius (or centred) representation [3]–[7], [15], [16]. In the centred representation, we consider the "nominal" value of an uncertain entity (midpoint or centre of the corresponding interval) together with an upper bound of its measurement error (radius of the interval). Arithmetical operations on intervals are defined for both representations, allowing to formulate and solve interval equations algebraically and numerically. Of these operations, multiplication is the most important and troublesome at the same time, as it is just the formula for it that is responsible for coupling the coordinates of the interval (endpoints, or midpoint and radius). Addition is done coordinate-wise in both representations, while division is equivalent to multiplication by a reciprocal, which in turn reduces to multiplication by a scalar (see Section 3.1). In subtraction and multiplication by scalars the coordinates are also separated in the centred representation, while in the endpoints representation the coordinates are generally not separated.

In the centred representation it is quite natural to think of multiplication of uncertain quantities in terms of first multiplying their nominal values, and then estimate the maximal error of the result from the errors of the arguments. This automatically separates the coordinates, but the result is, unfortunately, not equal to the true (called also "set-theoretic") interval multiplication—the centre of the true product of intervals is generally not equal to the product of centres of the multiplicands. One of the operations for which the centre of the product is equal to the product of the centres is the "centred outward" multiplication of intervals, proposed for the analysis and solution of linear systems with interval coefficients [7], [13], [14]. The centred outward interval multiplication uses a simpler formula and produces an *outer* enclosure for the set-theoretic interval product. From a practical point of view, it would thus be useful to have an analogical approximate multiplication operation that will produce an *inner* approximation of the true product. Using such a pair of operations we could solve the equations approximately and separately for each coordinate and obtain automatically the estimate of the maximal error of the approximation caused by the use of these simplified operations.

In the present work we have studied diagrammatically various types of multiplication of intervals—the standard set-theoretic interval multiplication, the "centred outward", or *co-multiplication*, and the new "centred inward", or *ci-multiplication* which we propose. The latter operation, suggested by the above considerations, has been properly defined with the help of the diagrammatic approach we use in the paper. Certain important properties of these multiplication operations have been studied and compared using the diagrammatic method. Most of all, we have studied estimation errors (we show that the centred operations approximate the standard product very tightly for narrow interval arguments, the most common in practical computations) and inclusion isotonicity properties (important for solving many types of interval equations).

Concerning the diagrammatic approach, various simple diagrams appeared in the literature on interval analysis, but as yet they played no significant role in the development of the field. They were neither systematically investigated, nor more widely applied in interval analysis and its applications. This stays in marked contrast to the development of complex number theory and analysis long ago, where the diagrammatic notation based on the complex plane diagram played an important role in the acceptance of complex numbers as legitimate mathematical objects and in the development of their theory. Even today, new capabilities of this notation are being discovered [9]. Therefore, we think that extending the use of diagrammatic methods to interval analysis might also prove fruitful and deserves some attention.

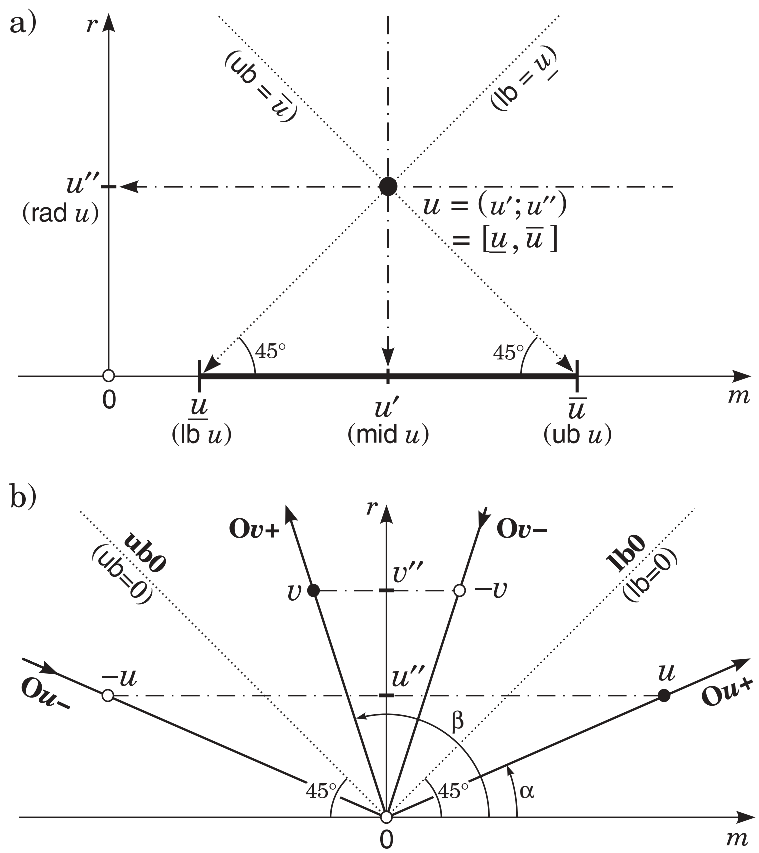


Figure 1.1: The MR-diagram: interval representation and graphical elements.

2 The MR-diagram and its basic uses.

Given $u', u'' \in \mathbb{R}$, $u'' \geq 0$, the set $u = \{\tilde{u} \,|\, u' - u'' \leq \tilde{u} \leq u' + u''\}$ is an *interval* on the real line $\mathbb{R}$ with *midpoint (centre)* u' and *radius* u''. The set of all intervals on $\mathbb{R}$ is denoted by $\mathbb{IR}$.

In this work we shall make use of the *MR-diagram* representation of intervals [3], [4], see Fig. 1.1. In a MR-diagram, the interval is represented by a point with its centred coordinates: midpoint u' and radius u''. Thus, any interval u is identified by the pair $u = (u'; u'')$, where $u', u'' \in \mathbb{R}$, $u'' \geq 0$. An interval of the form $u = (u'; 0)$ is called a thin (or point) interval, and an interval $v = (0; v'')$ is called zero-symmetric or just symmetric.

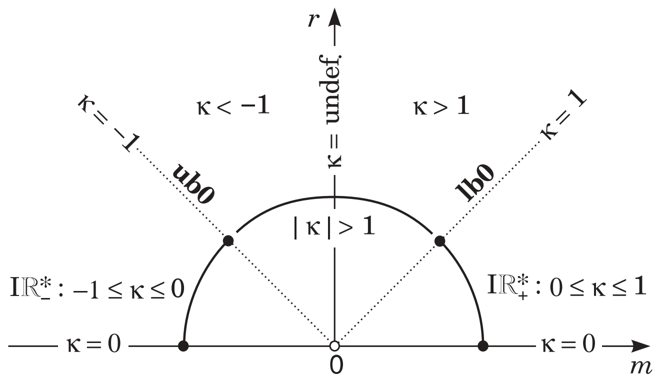


Figure 2.1: Regions in the MR-diagram with corresponding values of the κ function.

Besides midpoint and radius, one can also easily obtain the endpoints $\underline{u}$, $\overline{u}$ of the interval $u = [\underline{u}, \overline{u}]$ using the diagonal lines, Fig. 1.1a. In this way, the MR-diagram combines conveniently all three common representations of intervals — midpoint-radius, endpoint, and the one-dimensional representation as a segment of a real line (here on the **Om** axis).

The main diagonals **lb0** and **ub0** constitute a dividing line between intervals *containing zero* (they all lie on or above the diagonals) and those *without zero* (below the diagonals), see Fig. 1.1b and Fig. 2.1. The *interval axis*[1] $\mathbf{O}\boldsymbol{u}$ of the interval u consists of a positive half $\mathbf{O}\boldsymbol{u}+$ going through the interval u, and the negative half $\mathbf{O}\boldsymbol{u}-$ through the interval $-u$.

The *relative extent* function $\kappa(u) = u''/u'$ [4] (also called **rex** u) belongs to a family of functions of this type, the most popular being the χ function [12]. For the comparison and polar graphs of the functions κ and χ see [4]. As shown in Fig. 2.1, the value of $|\kappa(u)|$ is greater than or equal to 1 for intervals containing 0, and less than 1 for intervals without 0. Intervals without zero are positive if $0 \leq \kappa(u) < 1$, and negative if $-1 < \kappa(u) \leq 0$. For all zero-symmetric intervals $(0; r), r \geq 0$, lying on the **Or** axis (including the origin), we assume that $\kappa((0; r))$ is undefined. We call the interval v *more extended* than the interval u when $|\kappa(v)| > |\kappa(u)|$, i.e., when v lies above the axis $\mathbf{O}\boldsymbol{u}$ of the interval u on the MR-diagram (and hence, u lies below the axis $\mathbf{O}\boldsymbol{v}$ of v, cf. Fig. 1.1b). We also consider symmetric intervals distinct from 0 to be more extended that all nonsymmetric ones.

[1] In older papers and reports [3]–[5] this was called a "radial line." The term "interval axis" is more appropriate, as will become clear in the sequel.

Denote by $\mathbb{IR}^*$ the set of all intervals which do not contain zero as an interior point: $\mathbb{IR}^* = \{(u';u'') \in \mathbb{IR} \mid u'' \le |u'|\} = \{u \in \mathbb{IR} \mid |\kappa(u)| \le 1\} \cup \{0\}$.

Inclusion of intervals is expressed in the midpoint-radius form by [11], [16]:

$$u \subseteq v \iff |v' - u'| \le v'' - u'', \quad u, v \in \mathbb{IR}. \tag{2.1}$$

The diagrammatic representation of the above formula, as well as the regions in the MR-diagram containing intervals included in the interval u, including it, preceding it, and following it are shown in Fig. 2.2a. For a given interval relation $\diamondsuit$, we denote a *coimage* of an interval v under this relation as $(\diamondsuit v) = \{u \mid u \diamondsuit v\}$, e.g. $(\subseteq v) = \{u \mid u \subseteq v\}$.

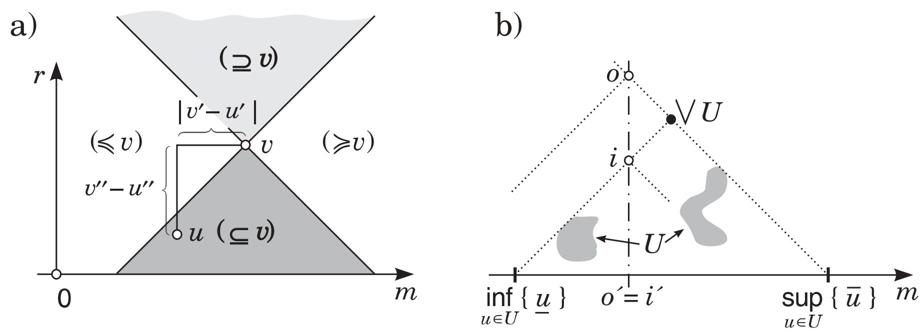


Figure 2.2: Diagrammatic illustration of interval inclusion (a), and inner and outer approximation of the interval hull (b).

The *interval hull* of a bounded set U of intervals (or reals) is defined as $\bigvee U = [\inf\{\underline{u} \mid u \in U\}, \sup\{\overline{u} \mid u \in U\}]$. For two intervals it constitutes the join operation $\vee$ of the lattice of intervals generated by the interval inclusion relation (2.1). It is constructed in the MR-diagram as an intersection of diagonal lines tightly embracing from the above the argument set, see Fig. 2.2b. As Fig. 1.1 thus shows, for any interval u we have $u = \bigvee\{\tilde{u} \mid \tilde{u} \in u\} = \underline{u} \vee \overline{u}$.

In Fig. 2.2b a construction is shown, to be used in the sequel, for the two intervals o and i with the given midpoint $o' = i'$ that constitute tightest outward and largest inward, respectively, approximations of the hull.

3 Interval multiplication.

3.1 Multiplication by scalars.

Multiplication of an interval u by a scalar (real number) $m \in \mathbb{R}$ is defined by $m \cdot u = \{m\tilde{u} \mid \tilde{u} \in u\} = (mu'; |m|u'')$. The *interval axis* groups all products of the interval u and all real numbers, symbolically: $\mathbf{O}\boldsymbol{u} = \mathbb{R} \cdot u$. Intervals lying on the same axis share many common properties, e.g., they have the same absolute value of the function κ (excluding the interval $[0,0]$ at the origin), i.e., $|\kappa(v)| = |\kappa(u)|$ for all $v \in \mathbf{O}\boldsymbol{u} \setminus \{[0,0]\}$.

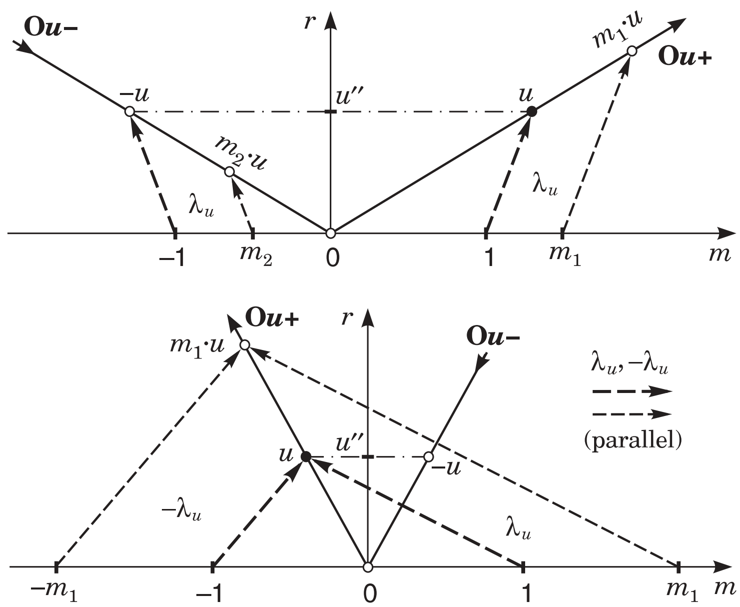


Figure 3.1: Multiplication by a number (*lambda mapping*) $\lambda_u(m) = m \cdot u$.

To find the product of an interval u and a real number m, it suffices to map appropriately the point on the **Om** axis with the coordinate m onto the interval axis $\mathbf{O}\boldsymbol{u}$. The diagrammatic constructions for that are shown in Fig. 3.1. It is convenient to define the mapping as a function called here *lambda mapping*: $\lambda_u(m) = m \cdot u$. Its inverse allows to find the real number (a point on the **Om** axis) by which the interval u has been multiplied to obtain the given point on the axis $\mathbf{O}\boldsymbol{u}$. Depending on the need, one may use either the mapping lines parallel to the lines from the points of value $+1$ and -1 to u and $-u$, respectively, or from $+1$ to $-u$ and from -1 to u, taking into account the equalities $\lambda_{-u}(m) = \lambda_u(-m) = -\lambda_u(m)$, see Fig. 3.1.

It is easy to observe inclusion isotonicity of multiplication by real scalars: for $u, v \in I\mathbb{R}$, $t \in \mathbb{R}$, $t \neq 0$, we have

$$v \subseteq u \Longleftrightarrow t \cdot v \subseteq t \cdot u. \tag{3.1}$$

See Fig. 3.2 for a diagrammatic demonstration of the property. As the segment connecting the points u and v is parallel to the segment connecting the points $t \cdot u$ and $t \cdot v$, the relation of inclusion between them, which depends on the direction of that segment, remains unchanged.

It is also important to note that obtaining a reciprocal $1/u$ of an interval u (for $0 \notin u$) reduces to multiplication of u by a scalar: $1/u = \gamma u$, where $\gamma^{-1} = (u')^2 - (u'')^2$. Thus, the reciprocal of u lies on the interval axis $\mathbf{O}\boldsymbol{u}$, see [4] for a diagrammatic construction for that.

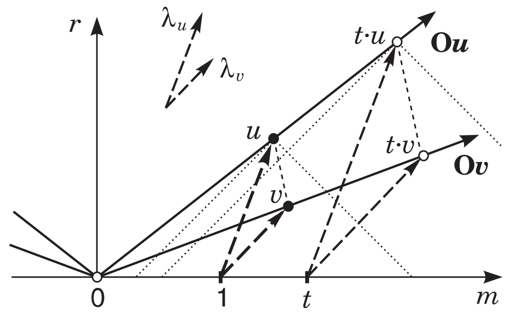


Figure 3.2: Inclusion monotonicity of multiplication by a number.

3.2 Set-theoretic interval multiplication.

With the above constructions ready, the product of two intervals can be constructed according to the formula, see also [4]:

$$u \cdot v = \{\tilde{u}\tilde{v} \mid \tilde{u} \in u,\, \tilde{v} \in v\} = \bigvee \{\tilde{v} \cdot u \mid \tilde{v} \in v\} = \underline{v} \cdot u \vee \overline{v} \cdot u. \tag{3.2}$$

Examples of the construction of the interval product (3.2) for typical cases are shown in Figs. 3.3 and 3.4, where the thick gray segment(s) of the $\mathbf{O}\boldsymbol{u}$ axis constitute the set $\{\tilde{v} \cdot u \mid \tilde{v} \in v\}$. The first step consists of drawing an interval axis of one of the intervals, say u, and then projecting the other interval, here v (in fact, it suffices to project its endpoints only) onto the axis by lambda mapping, as explained in Fig. 3.1, obtaining the points (intervals) $\underline{v} \cdot u$ and $\overline{v} \cdot u$. The interval hull of the resulting segments (thick gray segments: again, taking their endpoints suffices) gives the product $u \cdot v = \underline{v} \cdot u \vee \overline{v} \cdot u$.

Interval multiplication (3.2) is expressed by means of midpoint and radius as follows (the formula was given already by Warmus [16] and Sunaga [15]):

$$\begin{aligned} u \cdot v = (&u'v' + \min\{u''|v'|, |u'|v'', u''v''\} \cdot \operatorname{sign}(u'v'); \\ &u''|v'| + |u'|v'' + u''v'' - \min\{u''|v'|, |u'|v'', u''v''\}). \end{aligned} \tag{3.3}$$

For $u, v \in \mathbb{IR}^*$, the formula (3.3) obtains a simpler form:

$$(u'; u'') \cdot (v'; v'') = \begin{cases} (u'v' + u''v''; |v'|u'' + |u'|v''), & \text{if } u'v' \geq 0, \\ (u'v' - u''v''; |v'|u'' + |u'|v''), & \text{if } u'v' < 0. \end{cases} \tag{3.4}$$

When at least one of the arguments contains zero, and $|\kappa(u)| \geq |\kappa(v)|$, the formula obtains the form:

$$(u'; u'') \cdot (v'; v'') = \begin{cases} (u'(|v'| + v''); u''(|v'| + v'')), & \text{if } v' \geq 0, \\ (-u'(|v'| + v''); u''(|v'| + v'')), & \text{if } v' < 0. \end{cases} \tag{3.5}$$

When $|\kappa(u)| < |\kappa(v)|$, (3.5) is still valid after exchanging u and v.

As it is seen from (3.3)–(3.5), interval coordinates in interval equations that use the standard multiplication of intervals are coupled. It would be nice if the

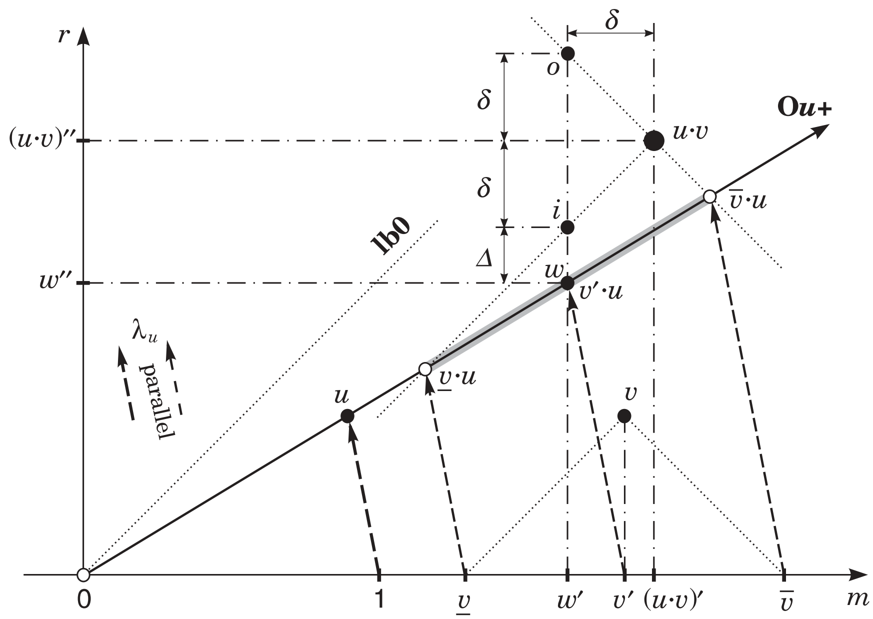


Figure 3.3: Standard interval multiplication, co-multiplication, and ci-multiplication compared (for positive intervals).

coordinates in the formulas were separated, i.e. if either the midpoint, or the radius of the result contained one type of coordinates only (midpoints or radii). However, the formulas for standard multiplication do not allow for separation of coordinates in formulas, both in endpoints and midpoint-radius coordinates (and in fact in any other pair of coordinates, as was shown by Ratschek [12]).

Since in interval computations we are dealing with approximations anyway, the idea arises of defining *approximate multiplication* operations that would, at least partially, lack the above deficiency (and hopefully lead to simpler formulas), while approximating the familiar interval multiplication (3.2) as closely as possible and retaining its basic properties.

3.3 Centred outer interval multiplication.

Consider the so-called *centred outer multiplication* (*co-multiplication*), defined as:

$$u \circ v = (u'v'; u''|v'| + |u'|v'' + u''v''). \tag{3.6}$$

It is readily seen that $u \cdot v \subseteq u \circ v$. Note that the midpoint of (3.6) is the product of only the *midpoints* of the arguments (it causes the interval coordinates to become decoupled), while the radius formula is simplified as well, in such a way that the result always *contains* the product (3.2), as indicated (by omitting the $\min\{\}$ component from the radius part of (3.3) the interval is actually widened).

A diagrammatic construction for the operation (3.6) is in order. Let us assume in the sequel that $|\kappa(u)| \geq |\kappa(v)|$ (otherwise, exchange u and v) and $u', v' > 0$ (other cases produce essentially the same constructions, only with some of its parts mirrored in the **Or** axis, see [3], [4]).

Let us start from the case of both u and v without zero. Then the construction in Fig. 3.3 is obtained as follows. First, find the point (interval) $u \cdot v = \underline{v} \cdot u \vee \overline{v} \cdot u$ as already described; second, find the middle point (interval) w of the thick gray segment (we obviously have $w = v' \cdot u$); and third, find the interval o with centre w' (that is the interval on the vertical line through w), which contains the thick gray segment (and hence contains the product $u \cdot v$) and has the smallest possible radius, cf. also Fig. 2.2b, taking the thick gray segment for U.

The following equations can be easily deduced from Fig. 3.3:

$$\begin{aligned}
&(u \cdot v)' = u'v' + u''v'',\\
&(u \cdot v)'' = u'v'' + u''v',\\
&w = v' \cdot u = (u'v'; u''v'),\\
&\delta = |(u \cdot v)' - w'| = u''v'',\\
&\Delta = |(u \cdot v)'' - w'' - \delta| = u''v' - u''v''.
\end{aligned}$$

Therefore:

$$u \circ v = (u'v'; u''v' + u'v'' + u''v'') = (u'v'; (u \cdot v)'' + \delta) = o.$$

Similarly, one may trace the construction of the co-product (3.6) in Fig. 3.4 for operands containing zero; the detailed analysis of this case is conducted in the next section.

3.4 Centred inner interval multiplication.

In Fig. 3.3 we see that the interval labelled i has the same midpoint as the interval o, while having the largest radius amongst the intervals *included* in the product (3.2), cf. also Fig. 2.2b. For positive intervals we thus arrive at the following formula for a new *centred inner multiplication* (*ci-multiplication*):

$$u \diamond v = (u'v'; u''v' + u'v'' - u''v'') = (u'v'; (u \cdot v)'' - \delta) = i \subseteq u \cdot v. \tag{3.7}$$

As can be seen in Fig. 3.3 and the formulas above, both centres of the co-product and ci-product are equal to the product of centres of u and v, while radii are larger or smaller, respectively, from the radius of the standard product $u \cdot v$ by δ (here the product of radii of u and v). In this way, midpoints of these products depend only on midpoints of the arguments (which allows for the separation of coordinates). Their radii are then modified by the same amount δ in order to assure the inclusion relations $u \diamond v \subseteq u \cdot v \subseteq u \circ v$. As a result, formulas become simpler than for the standard multiplication (3.3)–(3.5), coordinates become decoupled, and the two operations estimate the standard multiplication from outside and from within.

The construction shows clearly that the centred products can be defined uniformly as:

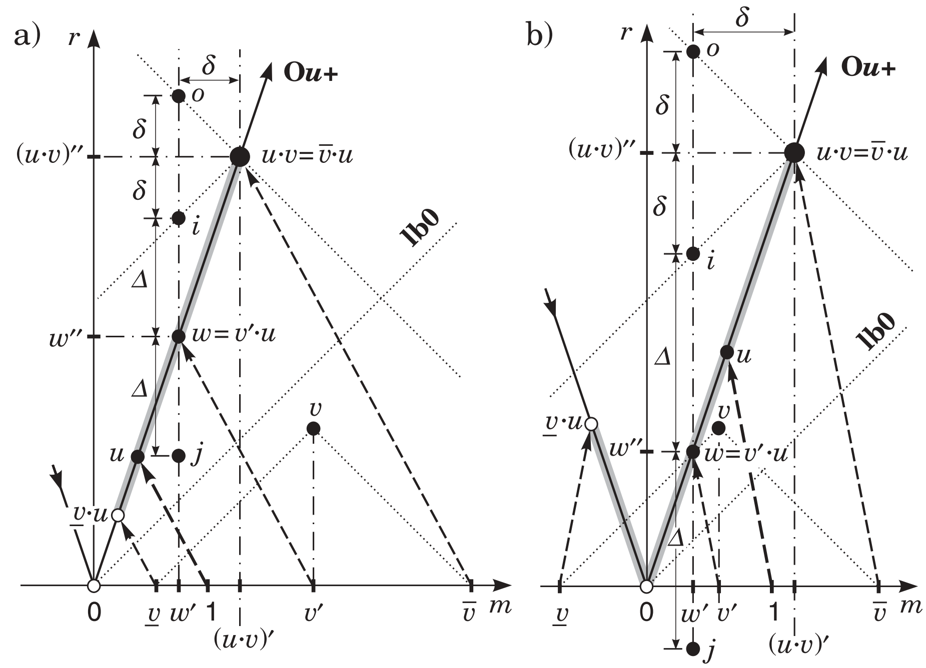


Figure 3.4: Interval multiplication operations when one (a) or both (b) operands contain zero.

Definition 3.1 (Co- and ci-multiplication). *For intervals u and v, their* co-product $u \circ v$ *is the narrowest interval containing $u \cdot v$ with midpoint equal to $u'v'$, while their* ci-product $u \diamond v$ *is the widest interval contained in $u \cdot v$ with the same midpoint.*

More formally, $u \circ v = (u'v'; R(u,v))$, where $R(u,v)$ is the smallest number such that $u \cdot v \subseteq u \circ v$, while $u \diamond v = (u'v'; r(u,v))$, where $r(u,v)$ is the largest number such that $u \diamond v \subseteq u \cdot v$.

Let us see how the above carries over to the case with one or both of the arguments containing zero. When only one of the arguments contains zero, the construction looks like that in Fig. 3.4a. Here we have:

$$\begin{aligned}
&(u \cdot v)' = (\overline{v} \cdot u)' = (v' + v'')u' = u'v' + u'v'',\\
&(u \cdot v)'' = (\overline{v} \cdot u)'' = (v' + v'')u'' = u''v' + u''v'',\\
&w = v' \cdot u = (u'v'; u''v'),\\
&\delta = |(u \cdot v)' - w'| = u'v'',\\
&\varDelta = |(u \cdot v)'' - w'' - \delta| = u''v'' - u'v''.
\end{aligned}$$

Intervals o and i have the same properties as described before for the co-product and the ci-product, respectively. However, only the co-multiplication formula (3.6) carries over well to this case, i.e. we have still:

$$u \circ v = (u'v'; u''v' + u'v'' + u''v'') = (u'v'; (u \cdot v)'' + \delta) = o,$$

whereas the formula (3.7) for ci-multiplication gives now different result, namely:

$$(u'v'; u''v' + u'v'' - u''v'') = (u'v'; w'' - \Delta) = j \neq i.$$

The formula for the interval i compatible with the Definition 3.1 now reads, as can be derived from the above construction:

$$i = (u'v'; (u \cdot v)'' - \delta) = (u'v'; u''v' - u'v'' + u''v'').$$

In the case of both u and v containing zero, the situation is similar, see Fig. 3.4b. The formulas for standard product $u \cdot v$, the w interval, the radial shifts δ and Δ, and the co-product $u \circ v$ remain the same as before. However, the first formula (3.7) for the ci-product would have to be changed anyway, as it can produce the negative radius $w'' - \Delta$, see Fig. 3.4b (that situation occurs when $1/|\kappa(u)| + 1/|\kappa(v)| < 1$). As a result, the proper formulas conforming to the Definition 3.1 for both approximate products are:

Proposition 3.1 (Co- and ci-multiplication). *General formulas for* co-multiplication $u \circ v$ *and* ci-multiplication $u \diamond v$, *valid for all types of interval arguments are (assuming* $|\kappa(u)| \geq |\kappa(v)|$*):*

$$\begin{aligned} u \circ v &= (u'v'; u''|v'| + |u'|v'' + u''v'') = (u'v'; (u \cdot v)'' + \delta), \\ u \diamond v &= (u'v'; u''|v'| + ||u'|v'' - u''v''|) = (u'v'; (u \cdot v)'' - \delta), \end{aligned} \tag{3.8}$$

where:

$$(u \cdot v)'' = \begin{cases} u''|v'| + |u'|v'' & \text{for } 0 \notin u, \\ u''|v'| + u''v'' & \text{for } 0 \in u. \end{cases} \tag{3.9}$$

$$\delta = min\{u'', |u'|\}v'' = \begin{cases} u''v'' & \text{for } 0 \notin u, \\ |u'|v'' & \text{for } 0 \in u. \end{cases} \tag{3.10}$$

In the case when $|\kappa(u)| < |\kappa(v)|$*, one should only exchange* u *and* v *in the above formulas—it would influence only the second formula (for ci-multiplication) in (3.8) and the case for* $0 \in u$ *in (3.9) and (3.10) (which would then change to* $0 \in v$*).*

4 Over- and underestimation of centred operations.

From a practical point of view it is important to be able to calculate the error introduced by the centred operations—overestimation of the true product produced by the co-multiplication operation, and underestimation produced by the ci-multiplication. The qualitative picture should be clear from the constructions already discussed (see Figs. 3.3 and 3.4). Figure 4.1 summarizes it more clearly: when the centre of the product is positive, co-multiplication overestimates the true product by the value 2δ at its lower bound, while ci-multiplication underestimates it by the same value 2δ at its upper bound. For a negative centre, the situation is simply mirrored in the **Or** axis. The midpoint of both products is uniformly shifted with respect to the standard product $u \cdot v$ by δ towards zero (formally, $u'v' = (u \cdot v)' - \delta \cdot \mathrm{sign}(u'v')$).

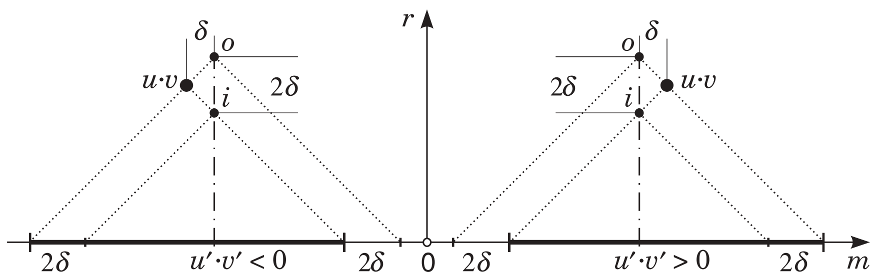
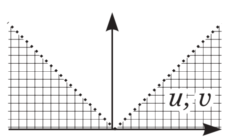


Figure 4.1: Over- and underestimation of outer and inner centred multiplication operations, for middle-negative and middle-positive products.

The value of δ is given by the formula (3.10) in Proposition 3.1. As can be seen, there are in fact three cases—the two listed explicitly in the proposition and the third one obtained by exchanging u and v in the case when $|\kappa(u)| < |\kappa(v)|$ and $0 \in u$ (exchanging u and v in the first case leads to the same formulas due to symmetry). For practice it is more important to calculate the *relative error*, with respect to the width of the true product, i.e., the quantity $d = 2\delta/2(u \cdot v)'' = \delta/(u \cdot v)''$. For both u and v without zero, from (3.9) and (3.10) we have thus $d = u''v''/(|v'|u'' + |u'|v'') = (u''v''/|u'||v'|)/(u''/|u'| + v''/|v'|) = |\kappa(u)||\kappa(v)|/(|\kappa(u)| + |\kappa(v)|)$. Using (3.9) and (3.10) for the case $0 \in u$ and $|\kappa(u)| \geq |\kappa(v)|$, we may derive directly the formula for d for this case as well. In the remaining third case, when $|\kappa(u)| < |\kappa(v)|$, one should only exchange u and v in the formula obtained above. In summary:

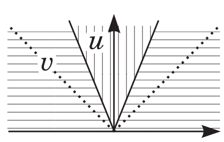
Case (a): $|\kappa(u)|, |\kappa(v)| \leq 1$, i.e., $u, v \in \mathbb{IR}^*$



$$\delta = u''v'',$$

$$d = \frac{|\kappa(u)||\kappa(v)|}{|\kappa(u)| + |\kappa(v)|} = (|\kappa(u)|^{-1} + |\kappa(v)|^{-1})^{-1}.$$

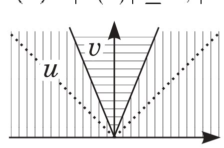
Case (b): $|\kappa(u)| \geq 1, |\kappa(u)| \geq |\kappa(v)|$



$$\delta = |u'|v'',$$

$$d = \frac{|\kappa(v)|}{|\kappa(u)|(1 + |\kappa(v)|)} = (|\kappa(u)|(1 + |\kappa(v)|^{-1}))^{-1}.$$

Case (c): $|\kappa(v)| \geq 1, |\kappa(v)| \geq |\kappa(u)|$



$$\delta = u''|v'|,$$

$$d = \frac{|\kappa(u)|}{|\kappa(v)|(1 + |\kappa(u)|)} = (|\kappa(v)|(1 + |\kappa(u)|^{-1}))^{-1}.$$

As it is seen in the above formulas, the relative error d depends only on the absolute value of the relative extent function κ of u and v. Thus, the error

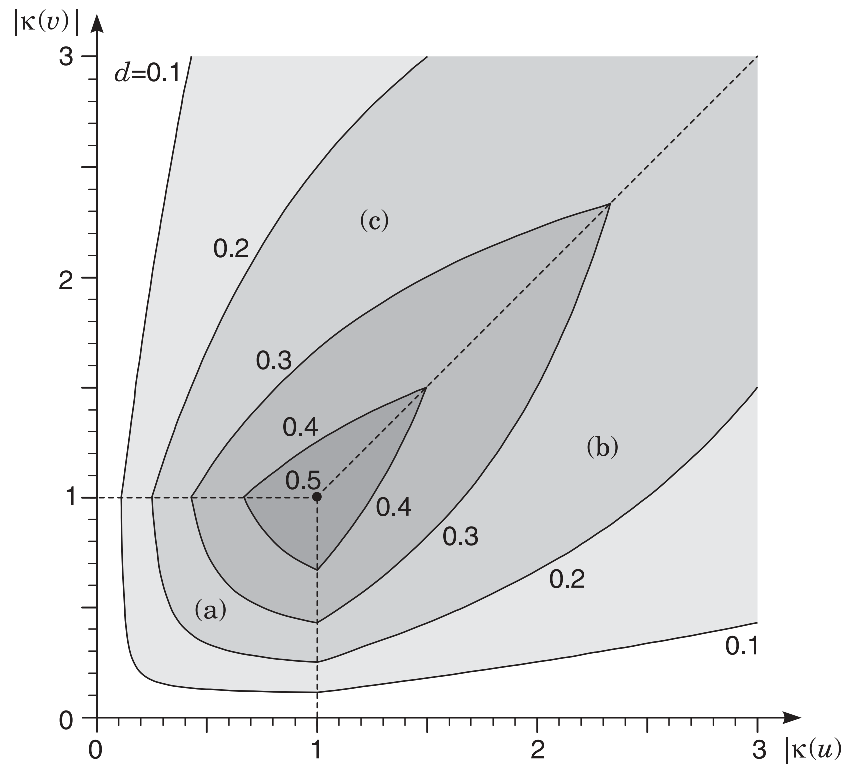


Figure 4.2: Isoline graph for the relative estimation error $d = \delta/(u \cdot v)''$ as function of relative extents $|\kappa(u)|$ and $|\kappa(v)|$.

function d can be plotted on a diagram with $|\kappa(u)|$ and $|\kappa(v)|$ as coordinates (the so-called *|RR|-diagram*, a variant of the *RR-diagram* introduced in [6]). The isoline plot of d is given in Fig. 4.2. As can be seen, the maximum error equals 0.5 and is obtained for $|\kappa(u)| = |\kappa(v)| = 1$, i.e., for intervals having zero as one of their endpoints. The error decreases when the arguments become near to thin intervals (real numbers) or to symmetric intervals. When at least one of the arguments is thin or symmetric, the error vanishes, signifying that $u \cdot v = u \circ v = u \diamond v$ in such a case, see Proposition 5.1 below.

The above can be further illustrated by several numerical examples, provided in Table 4.1. The first example takes two intervals starting at zero. The absolute error δ equals here half the radius of the true product, thus $d = 0.5$ and is maximal. The second example takes two rather wide intervals, one containing zero inside and quite near to being symmetric. The error is much smaller—only 17%. In the third example, the zero-containing argument is replaced with comparatively narrow positive interval. The product is narrower than in the previous case, so with the same absolute error δ the relative error becomes

Table 4.1: Numerical examples

u	v	$\|\kappa(u)\|$	$\|\kappa(v)\|$	$u \cdot v$	$u \circ v$	$u \diamond v$	δ	d
$(1;1)$ $[0,2]$	$(3;3)$ $[0,6]$	1	1	$(6;6)$ $[0,12]$	$(3;9)$ $[-6,12]$	$(3;3)$ $[0,6]$	3	1/2 50%
$(2;1)$ $[1,3]$	$(-1;2)$ $[-3,1]$	1/2	2	$(-3;6)$ $[-9,3]$	$(-2;7)$ $[-9,5]$	$(-2;5)$ $[-7,3]$	1	1/6 17%
$(2;1)$ $[1,3]$	$(3;1)$ $[2,4]$	1/2	1/3	$(7;5)$ $[2,12]$	$(6;6)$ $[0,12]$	$(6;4)$ $[2,10]$	1	1/5 20%
$(50;1)$ $[49,51]$	$(50;1)$ $[49,51]$	1/50	1/50	$(2501;100)$ $[2401,2601]$	$(2500;101)$ $[2399,2601]$	$(2500;99)$ $[2401,2599]$	1	1/100 1%
$(50;1)$ $[49,51]$	$(1;50)$ $[-49,51]$	1/50	50	$(51;2550)$ $[-2499,2601]$	$(50;2551)$ $[-2500,2601]$	$(50;2549)$ $[-2499,2599]$	1	1/2550 0.04%

greater. Now we take a more practical example of rather narrow intervals, far more common in actual computations. With relative width of arguments of only 2%, the error of the centred operations drops to only 1% (the error would be only 0.04% in relation to the midpoint of the product). In the last example, one of the arguments is exchanged for a comparatively wide interval, very close to being symmetric. Now the product becomes very wide and the error drops even more, to only 0.04% (but as much as 2% relatively to the midpoint of the product).

5 Inclusion isotonicity properties.

In this section we study some properties of the multiplication operations that are related to the practically important inclusion relation.

As it is easily seen in the diagrams (Fig. 3.3 and Fig. 3.4, see also [4]), for every $u, v \in I\mathbb{R}$ we have $|\kappa(u \cdot v)| \geq |\kappa(u)|$, i.e., the product of two intervals is always more (or at least equally) extended than any of the arguments (see Section 2).

Moreover, when the more extended argument, say u, contains zero, i.e., $|\kappa(u)| \geq 1$ and $|\kappa(u)| \geq |\kappa(v)|$, then $u \cdot v$ lies on the interval axis of u, i.e., is a multiple of u by some real number (one of the endpoints of v: this is the case of a "simplified multiplication", see (3.5)). Now this property obviously does not hold for centred multiplication operations.

PROPOSITION 5.1 (THIN OR SYMMETRIC ARGUMENTS). *The following properties hold:*

1. *If one of the arguments is thin (i.e., lies on the* $\boldsymbol{Om}$ *axis), then* $u \cdot v = u \circ v = u \diamond v$ *and the product lies on the interval axis of the second argument.*

2. *If one of the arguments is symmetric (i.e., lies on the* $\boldsymbol{Or}$ *axis), then* $u \cdot v = u \circ v = u \diamond v$ *and the product is also symmetric.*

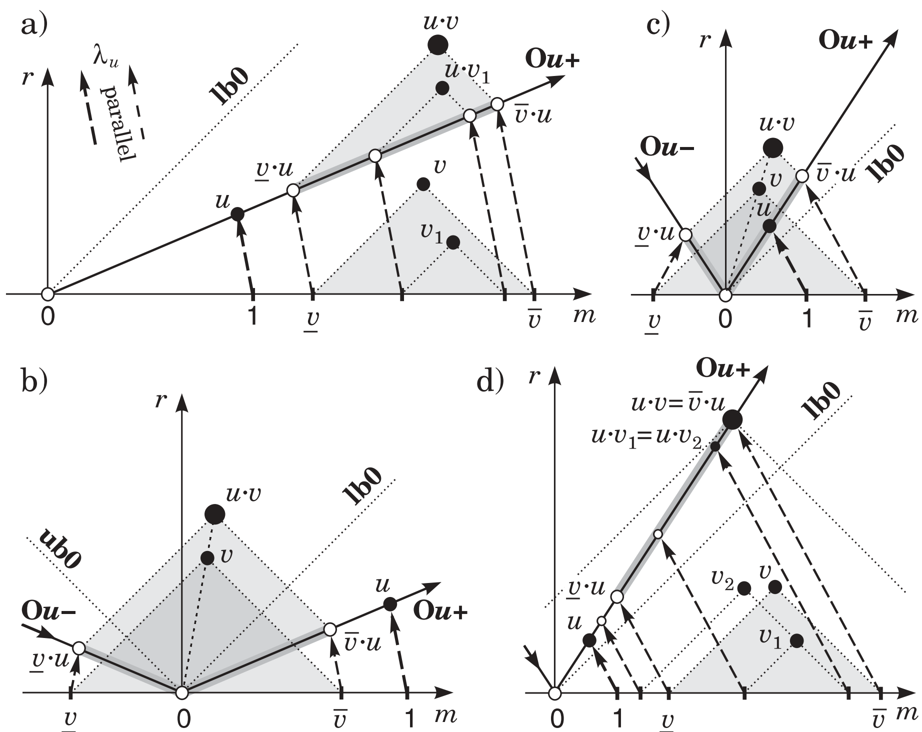


Figure 5.1: Inclusion monotonicity and inverse inclusion monotonicity hold for cases (a, b, c), because interval multiplication defines a bijection between the two gray sets (two triangles or triangle and quadrangle; darker gray marks the intersection of the sets); in case (d) only normal inclusion monotonicity holds and the mapping ceases to be a bijection.

The above properties are easy to show from the formulas for the multiplication operations defined above, or demonstrated with diagrams (for the second property, consider especially Fig. 5.2c and Fig. 5.3c and d).

The following proposition follows easily from (3.1) and (3.2) and is well known:

Proposition 5.2 (Inclusion monotonicity). *For* $v, v_1, u \in \mathbb{IR}$,

$$v_1 \subseteq v \Longrightarrow u \cdot v_1 \subseteq u \cdot v. \tag{5.1}$$

See Fig. 5.1a for the demonstration in the case of positive intervals ($u, v > 0$). For negative intervals the construction is (fully or in part) mirror-reflected in the **Or** axis. For intervals containing zero, Figs. 5.1b-d demonstrate the property.

The same diagrams demonstrate the inverse inclusion isotonicity of interval multiplication for proper intervals, that is (5.1) in the inverse direction.

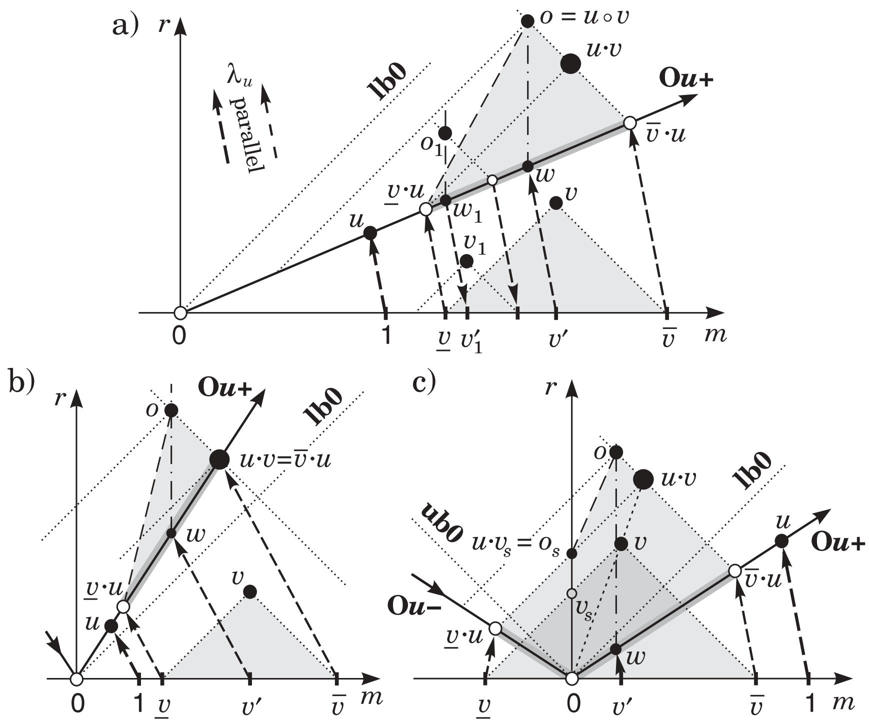


Figure 5.2: Inclusion monotonicity of the co-multiplication holds for all cases (a, b, c), due to the bijection between the two gray regions (darker gray marks intersection of the regions); however, the inverse isotonicity does not hold here (see text). The omitted case of both u and v containing zero is qualitatively the same as in (c).

Proposition 5.3 (Inverse inclusion monotonicity). *If* $v, v_1, u \in I\mathbb{R}$, *and either* $0 \notin u$ *or* $|\kappa(v)| \geq |\kappa(u)|$, *then*

$$u \cdot v_1 \subseteq u \cdot v \Longrightarrow v_1 \subseteq v. \tag{5.2}$$

A diagrammatic demonstration of the above is shown in Fig. 5.1a for positive intervals, in Fig. 5.1b when $0 \in v$, and in Fig. 5.1c when $|\kappa(v)| \geq |\kappa(u)| \geq 1$ (i.e. when both u and v contain zero and v is more extended than u). When $0 \in u$ but $|\kappa(v)| < |\kappa(u)|$, the property does not necessarily hold, as shown by the interval v_2 in Fig. 5.1d. Note that the formula (5.2) is not symmetric with respect to u and v, hence it is not the same whether the interval after the "$\Longrightarrow$" in the formula is more or less extended than the other. Therefore, one cannot exchange u with v and then Fig. 5.1b with Fig. 5.1d in the diagrammatic argument above. A much complicated algebraic proof of (5.2) under the more restrictive condition $0 \notin u$ is given in [7].

PROPOSITION 5.4 (ISOTONICITY OF CO-MULTIPLICATION). *For $v, v_1, u \in \mathbb{IR}$, we have*

$$v_1 \subseteq v \Longrightarrow v_1 \circ u \subseteq v \circ u. \tag{5.3}$$

The property is illustrated in Fig. 5.2 (compare with Fig. 5.1; only here in Fig. 5.2b we have the case $0 \in u, 0 \notin v$, while in Fig. 5.2c the case $0 \notin u, 0 \in v$). It was also proven algebraically in [7]. Since the gray region above the $\mathbf{O}\boldsymbol{u}+$ axis in Fig. 5.2, containing results of co-multiplication of $(\subseteq v)$ by u, does not fill the whole triangle above $\mathbf{O}\boldsymbol{u}+$ included in $o = u \circ v$, the inverse isotonicity (i.e., the property $v_1 \circ u \subseteq v \circ u \Longrightarrow v_1 \subseteq v$) for the co-multiplication does not hold. It is explicitly shown in Fig. 5.2a where despite that $o_1 \subseteq v \circ u$ (where $o_1 = v_1 \circ u$) we have $v_1 \nsubseteq v$.

A somewhat reverse situation takes place for ci-multiplication: for it inverse inclusion monotonicity holds, but not the normal one.

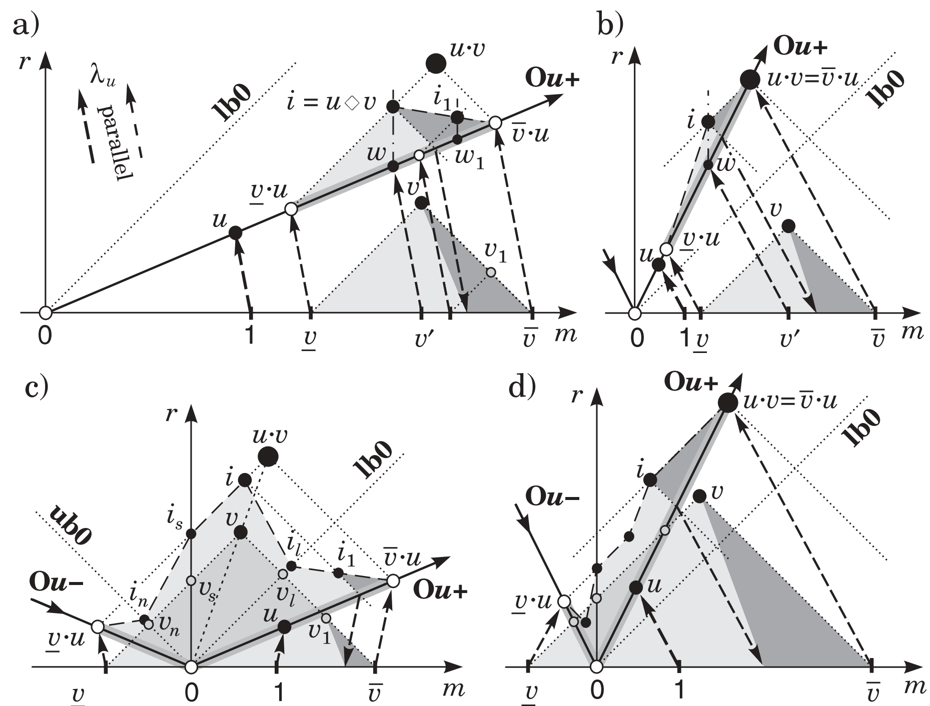


Figure 5.3: Inverse inclusion monotonicity of the ci-multiplication holds only for both u and v in $\mathbb{IR}^*$ (a), due to the bijection between the two light- and dark-gray regions; while no kind of isotonicity holds in other cases (see text). The omitted case of both u and v containing zero and v more extended than u is qualitatively the same as in (c).

PROPOSITION 5.5 (INVERSE ISOTONICITY OF CI-MULTIPLICATION). *For* $v, v_1, u \in \mathbb{IR}^*$, *we have*

$$v_1 \diamond u \subseteq v \diamond u \Longrightarrow v_1 \subseteq v. \tag{5.4}$$

The property is illustrated in Fig. 5.3a (compare with Fig. 5.2a). The property does not hold for u or v containing zero—as shown in Fig. 5.3b-d for all relevant cases (compare also with previous figures), there are regions above the $\mathbf{O}\boldsymbol{u}$ axis included in $i = u \diamond v$ that do not belong to the set of products of $(\subseteq v)$ by u (compare with Fig. 5.2). In Fig. 5.3c, appropriate indices $k \in \{n, s, l, 1\}$ show the correspondence between products i_k at the vertices of the gray region (containing the results of $(\subseteq v) \diamond u$) and the intervals $v_k \in v$ (gray-filled circles). In Fig. 5.3d the labels are omitted for clarity; it is assumed that the correspondence is easy to infer visually. Most of the λ-mapping constructions were also omitted for clarity.

Moreover, the region above the $\mathbf{O}\boldsymbol{u}+$ axis in Fig. 5.3 which contains results of ci-multiplication $(\subseteq v) \diamond u$ usually includes also a region outside the $(\subseteq i)$ triangle (the outside region being here marked with dark gray), that is, it contains intervals that are not included in $i = u \diamond v$. Therefore, the normal inclusion isotonicity (i.e., the property $v_1 \subseteq v \Longrightarrow v_1 \diamond u \subseteq v \diamond u$) for the ci-multiplication does not hold in general, as for some v_1 we have $i_1 = v_1 \diamond u \not\subseteq v \diamond u$. The only exceptions occur for thin and symmetric intervals, as follows from Proposition 5.1.

As can be seen from the analysis of conditions in Propositions 5.3-5.5 and accompanying discussion, the set of possible combinations of types of intervals u and v divides into three cases with respect to isotonicity properties, actually the same three cases as those considered before in Section 4. Thus, the inclusion isotonicity properties of set-theoretic and centred interval multiplication operations can be summarized, for all types of intervals, as in Fig. 5.4.

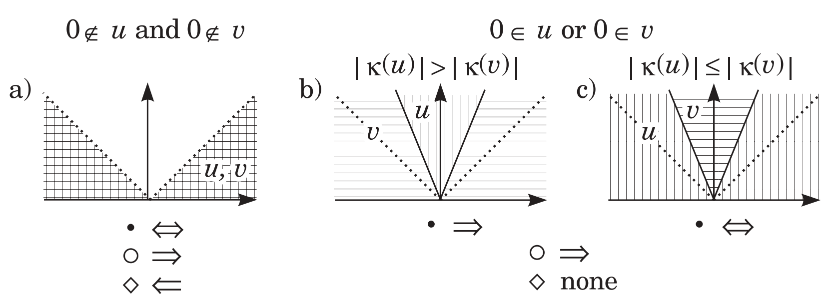


Figure 5.4: Inclusion isotonicity properties of set-theoretic $(\cdot)$, and centred multiplication operations: outer $(\circ)$ and inner $(\diamond)$ for all combination of argument types (a, b, c). The symbol $\Rightarrow$ indicates normal isotonicity only, $\Leftarrow$ inverse isotonicity only, and $\Leftrightarrow$ both kinds.

6 Notes and conclusions.

The centred outer interval multiplication (3.6) has been proposed in [16]; independently it has been introduced and practically implemented in [15], cf. [8]. Later on [10] it has been studied with respect to distributivity. As noted in [10], the operation (3.6) is a special case of the complex disc multiplication introduced in [1]. A software implementation of the co-multiplication is reported in [13]. The *overestimation* of co-multiplication was first estimated by the inequality $(u \circ v)''/(u \cdot v)'' \leq 1.5$ in [14]. Inclusion monotonicity and other properties of the centred outer interval multiplication, especially concerning its usefulness for solving some classes of interval linear equations, are studied in [7].

The centred operations have simpler defining formulas and allow for separation of coordinates, thus leading to simpler solutions of interval linear systems, as was shown in [7] for co-multiplication applied to certain types of such problems. The issue deserves a more detailed study and will be a subject of our further research. At the same time, they approximate the set-theoretic multiplication fairly tightly, especially for narrow intervals which are the most common in practical applications.

It is interesting to note that the diagrammatic approach turned out to be useful in finding a suitable definition of centred inner interval multiplication. This approach has been also helpful in finding proper conditions and properties of inclusion isotonicity of set-theoretic and centred multiplication operations. As an example, a substantial generalization of conditions for inverse inclusion isotonicity of set-theoretic multiplication has been obtained (Proposition 5.3), not speaking about a much simpler diagrammatic demonstration of the property than the complicated algebraic proof in [7].

The diagrammatical approach has also an evident methodological value. Let us only point out that it provides a visual explanation why centred outward multiplication is not inverse isotone, why centred inward multiplication is not isotone, etc., not to mention the usefulness of visualization of many important constructions, such as various types of interval multiplication.

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REFERENCES

1. P. Henrici, *Circular arithmetic and the determination of polynomial zeroes*, in Lecture Notes in Mathematics 228, J. L. Morris, ed., Springer-Verlag, Berlin, 1971, pp. 86–92.
2. Z. Kulpa, *Diagrammatic representation and reasoning*, Machine GRAPHICS & VISION, 3 (1994), pp. 77–103.
3. Z. Kulpa, *Diagrammatic representation of interval space: Part I: Basics; Part II: Arithmetic*, Internal Report, IFTR PAS, Warsaw, 1999.

4. Z. Kulpa, *Diagrammatic representation for interval arithmetic*, Linear Algebra Appl., 324 (2001), pp. 55–80.
5. Z. Kulpa, *Towards diagrammatic analysis of systems of interval "linear equations,"* in Scientific Computing, Validated Numerics, Interval Methods, W. Krämer, J. W. von Gudenberg, eds., Kluwer Academic/Plenum Publishers, New York, 2001, pp. 115–126.
6. Z. Kulpa, (2003) *Diagrammatic analysis of interval linear equations. Part I: Basic notions and the one-dimensional case*, Reliable Computing, 9 (2003), pp. 1–20.
7. S. Markov, *Computation of algebraic solutions of interval systems via systems of coordinates*, in Scientific Computing, Validated Numerics, Interval Methods, W. Krämer, J. W. von Gudenberg, eds., Kluwer Academic/Plenum Publishers, New York, 2001, pp. 103–114.
8. S. Markov, K. Okumura, (1999) *The contribution of T. Sunaga to interval analysis and reliable computing*, in Developments in Reliable Computing, T. Csendes, ed., Kluwer Academic, New York, 1999, pp. 167–188.
9. T. Needham, *Visual Complex Analysis*, Clarendon Press, Oxford, 1997.
10. A. Neumaier, *A distributive interval arithmetic*, Freiburger Intervall-Berichte, 82/10 (1982), pp. 31–38.
11. A. Neumaier, *Interval Methods for Systems of Equations*, Cambridge University Press, Cambridge, 1990.
12. H. Ratschek, *Representation of interval operations by coordinates*, Computing, 24 (1980), pp. 93–96.
13. S. M. Rump, *INTLAB — INTerval LABoratory*, in Developments in Reliable Computing, T. Csendes, ed., Kluwer Academic, New York, 1999, pp. 77–104.
14. S. M. Rump, *Fast and parallel interval arithmetic*, BIT, 39 (1999), pp. 534–554.
15. T. Sunaga, *Theory of an interval algebra and its application to numerical analysis*, RAAG Memoirs, 2(Misc. II) (1958), pp. 547–564.
16. M. Warmus, *Calculus of approximations*, Bull. Acad. Polon. Sci., Cl. III, IV (1956), pp. 253–259. [See `http://www.ippt.gov.pl/~zkulpa/quaphys/warmus.html#56`]