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On quasilinear spaces of convex bodies and intervals

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Abstract

Algebraic systems abstracting properties of convex bodies and intervals, with respect to addition and multiplication by scalars, known as quasilinear spaces, are studied axiomatically. We discuss special quasilinear spaces with group structure called quasivector spaces. We show that every quasivector space is a direct sum of a vector space and a symmetric quasivector space. A complete characterization of symmetric quasivector spaces in the finite dimensional case is given, which permits to reduce computation in quasilinear spaces to computation in familiar vector spaces.

Keywords: Convex bodies; Intervals; Vector space; Quasilinear space; Quasivector space; Symmetric quasivector space

1. Introduction

The set-theoretic operations for addition (so-called vector or Minkowski addition) and multiplication by scalars play important roles in convex and interval analysis and related applied problems, as natural arithmetic operations for convex bodies, resp. intervals. The abstract axiomatic study of these operations leads to the concept of quasilinear space [5,6,10,15,16]. A quasilinear space over the field of reals can be defined as an additive abelian monoid with cancellation law endowed with multiplication by scalars obeying the familiar four relations of vector spaces where the second distributive law is required to hold only for nonnegative scalars: $(\alpha+\beta)*C=\alpha*C+\beta*C$, if $\alpha\beta \geqslant 0$, cf. e.g. [17]. To see that the distributive law is violated when $\alpha\beta<0$, recall that a symmetric convex body C satisfies $(-1)*C=C$; hence $1*C+(-1)*C=C+C=2*C$. On the other side,

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$(1-1)*C=0*C=0$, showing that the second distributive law $(\alpha+\beta)*C=\alpha*C+\beta*C$ is not valid for $\alpha=1, \beta=-1$ (unless $C=0$).

As an abelian monoid with cancellation, a quasilinear space can be naturally embedded in an (additive) group; thereby an isomorphic extension of the multiplication by scalars leads to a special type of quasilinear spaces (those with group structure), here called quasivector spaces. Quasivector spaces obey all axioms of vector spaces, but a different distributivity relation corresponding to the second distributive law, namely: $(\alpha+\beta)*c=\alpha*c+\beta*c$, if $\alpha\beta \geqslant 0$. This relation can be formulated in a form convenient for symbolic computations, see Theorem 1; here we give a new simple proof of the latter relation. We also formulate and prove one more useful distributive relation (Theorem 2).

A main result in this work (Theorem 5) states that every quasivector space is a direct sum of a vector space and a symmetric quasivector space. This result has been proved in [7] in the finite dimensional case, here we prove it in the general case using a direct and simple approach. The main simplification in this approach consists in the introduction of basis only in symmetric quasivector spaces and not in general quasivector spaces, as in [7]. This leads to no loss of generality and makes the exposition very simple and straightforward. It has been also demonstrated that symmetric quasivector spaces are equivalent to vector spaces in the sense that the operations in both spaces are mutually interchangeable. This equivalence enables us to transfer basic concepts of vector spaces to symmetric quasivector spaces. We also prove that symmetric quasivector spaces with finite basis are isomorphic to a special canonic space.

In Section 2 we introduce some notation and give some examples of quasivector spaces, such as spaces of generalized convex bodies and intervals. In Section 3 we discuss calculation in quasivector spaces. Section 4 is devoted to the link between quasivector and vector spaces, as well as to the presentation of a quasivector space as a direct sum of a vector and a symmetric quasivector space. The latter spaces are characterized in Section 5.

2. Quasivector spaces

2.1. Quasilinear spaces

By $\mathbb{R}$ we denote the set of reals; we use the same notation for the linearly ordered (l.o.) field of reals $\mathbb{R}=(\mathbb{R},+,\cdot,\leqslant)$. Throughout the paper $\mathbb{R}$ can be replaced by any other linearly ordered field. For any integer $n \geqslant 1$ we denote by $\mathbb{R}^n$ the set of all n-tuples $(\alpha_1,\alpha_2,\ldots,\alpha_n)$, where $\alpha_i \in \mathbb{R}$. The set $\mathbb{R}^n$ forms a vector space under the operations of addition and multiplication by scalars denoted by $\mathbb{V}^n=(\mathbb{R}^n,+,\mathbb{R},\cdot)$, $n \geqslant 1$.

We recall that every abelian monoid $(\mathfrak{M},+)$ with cancellation law induces an abelian group $(\mathbb{D}(\mathfrak{M}),+)$, where $\mathbb{D}(\mathfrak{M})=\mathfrak{M}^2/\sim$ is the *difference* (*quotient*) *set* of $\mathfrak{M}$ consisting of all pairs (A,B) factorized by the congruence relation $\sim:(A,B)\sim(C,D)$ iff $A+D=B+C$, for all $A,B,C,D\in\mathfrak{M}$. Addition in $\mathbb{D}(\mathfrak{M})$ is defined by $(A,B)+(C,D)=(A+C,B+D)$. The neutral (null) element of $\mathbb{D}(\mathfrak{M})$ is the class (Z,Z), $Z\in\mathfrak{M}$; due to the existence of null element in $\mathfrak{M}$, we have $(Z,Z)\sim(0,0)$. The opposite element to $(A,B)\in\mathbb{D}(\mathfrak{M})$ is $\mathrm{opp}(A,B)=(B,A)$. The mapping $\varphi:\mathfrak{M}\to\mathbb{D}(\mathfrak{M})$ defined for $A\in\mathfrak{M}$ by $\varphi(A)=(A,0)\in\mathbb{D}(\mathfrak{M})$ is an *embedding* of monoids. We *embed* $\mathfrak{M}$ in $\mathbb{D}(\mathfrak{M})$ by identifying $A\in\mathfrak{M}$ with the equivalence class $(A,0)\sim(A+X,X)$, $X\in\mathfrak{M}$; all elements of $\mathbb{D}(\mathfrak{M})$ admitting the form $(A, 0)$ are called *proper* and the remaining are *improper*. The set of all proper elements of $\mathbb{D}(\mathfrak{M})$ is $\varphi(\mathfrak{M}) = \{(A, 0) \mid A \in \mathfrak{M}\} \cong \mathfrak{M}$.

Definition 1. Let $(\mathfrak{M}, +)$ be an abelian monoid with cancellation law. Assume that a mapping (multiplication by scalars) "$*$" is defined on $\mathbb{R} \times \mathfrak{M}$ satisfying:

(i) $\gamma * (A + B) = \gamma * A + \gamma * B$,
(ii) $\alpha * (\beta * C) = (\alpha\beta) * C$,
(iii) $1 * A = A$,
(iv) $(\alpha + \beta) * C = \alpha * C + \beta * C$, if $\alpha\beta \geqslant 0$.

The algebraic system $(\mathfrak{M}, +, \mathbb{R}, *)$ is called a (*cancellative*) *quasilinear space over* $\mathbb{R}$.

Remark. Definition 1 can be found in various modifications [5,6,10,12,13,15,16]. The condition $\alpha\beta \geqslant 0$ in the last axiom of the above definition is characteristic for convex bodies, cf. [17]. The isomorphism between (classes of) convex bodies and quasilinear spaces is studied in [16] (there the notion of $\mathbb{R}$-semigroup with cancellation law has been used).

Since $(\mathfrak{M}, +)$ is not assumed to be a group, there is no opposite in $(\mathfrak{M}, +, \mathbb{R}, *)$ in general, that is, for some $A \in \mathfrak{M}$ the equation $A + X = 0$ may not have a solution X. The operator $\neg A = (-1) * A$ is called *negation*. We write $A \neg B = A + (\neg B)$; note that $A \neg A = 0$ may not generally hold.

An element $A \in \mathfrak{M}$, such that $A \neg A = 0$, is called *linear*; in such case $\mathrm{opp}(A) = \neg A$. We denote $\mathfrak{M}' = \{A \in \mathfrak{M} \mid A \neg A = 0\}$. An element $A \in \mathfrak{M}$, such that $\neg A = A$, is called (*centrally*) *symmetric*; we denote $\mathfrak{M}'' = \{A \in \mathfrak{M} \mid \neg A = A\}$.

Using the above mentioned group extension method every quasilinear space $(\mathfrak{M}, +, \mathbb{R}, *)$ can be embedded into the group $(\mathbb{D}(\mathfrak{M}), +)$. Multiplication by scalars "$*$" is extended from $\mathbb{R} \times \mathfrak{M}$ to $\mathbb{R} \times \mathbb{D}(\mathfrak{M})$ by means of the following natural definition of $* : \mathbb{R} \times \mathbb{D}(\mathfrak{M}) \to \mathbb{D}(\mathfrak{M})$:

$$\gamma * (A, B) = (\gamma * A, \gamma * B), \quad A, B \in \mathfrak{M}, \ \gamma \in \mathbb{R}. \tag{1}$$

In particular, multiplication by the scalar -1 in $\mathbb{D}(\mathfrak{M})$, called *negation*, is

$$\neg(A, B) = (-1) * (A, B) = (\neg A, \neg B), \quad A, B \in \mathfrak{M}. \tag{2}$$

Note that (A, B) is proper if and only if $\gamma * (A, B)$ is proper. Indeed, $(A, B) = (C, 0) \Leftrightarrow A = B + C \Leftrightarrow \gamma * A = \gamma * (B + C) = \gamma * B + \gamma * C \Leftrightarrow (\gamma * A, \gamma * B) = (\gamma * C, 0)$.

Remark. Radström [14] studies the following multiplication by scalars in $(\mathbb{D}(\mathfrak{K}), +)$:

$$\gamma \cdot (A, B) = \begin{cases} (\gamma * A, \gamma * B), & \text{if } \gamma \geqslant 0, \\ (|\gamma| * B, |\gamma| * A), & \text{if } \gamma < 0. \end{cases} \tag{3}$$

If $\gamma < 0$, then $\gamma \cdot (A, 0) = (0, |\gamma| * A)$, which is an improper result. Therefore definition (3) does not provide an extension of multiplication by scalars in $\mathfrak{M}$.

More about quasilinear spaces with monoid structure can be found in [7]. Here we shall concentrate on the space $\mathbb{D}(\mathfrak{M})$, resp. on quasilinear spaces with group structure. In the sequel we shall use lower case roman letters to denote the elements of quasilinear spaces of group structure, such as

$\mathbb{D}(\mathfrak{M})$, writing e.g. $a=(A_1,A_2), A_1,A_2\in\mathfrak{M}$. For example, (2) can be written: $\neg a=(-1)*a$; $a\neg b$ means $a+(\neg b)$, etc.

2.2. Quasivector spaces: definition

Definition 2. A *quasivector space* (over the l.o. field $\mathbb{R}$), denoted $(\mathbb{Q},+,\mathbb{R},*)$, is an abelian group $(\mathbb{Q},+)$ with a mapping (multiplication by scalars) "$*$": $\mathbb{R}\times\mathbb{Q}\to\mathbb{Q}$, such that for $a,b,c\in\mathbb{Q}$, $\alpha,\beta,\gamma\in\mathbb{R}$:

$$\gamma*(a+b)=\gamma*a+\gamma*b, \tag{4}$$

$$\alpha*(\beta*c)=(\alpha\beta)*c, \tag{5}$$

$$1*a=a, \tag{6}$$

$$(\alpha+\beta)*c=\alpha*c+\beta*c, \quad \text{if } \alpha\beta\geqslant 0. \tag{7}$$

Remarks. (1) In (5) and (7) the sum $\alpha+\beta$, resp. the product $\alpha\beta=\alpha\cdot\beta$ and the relation $\alpha\beta\geqslant 0$ are well-defined in the l.o. real field $\mathbb{R}=(\mathbb{R},+,\cdot,\leqslant)$.

(2) It is easy to see that, if the condition $\alpha\beta\geqslant 0$ in the *quasidistributive law* (7) is replaced by the condition $\alpha\geqslant 0$, $\beta\geqslant 0$, then an equivalent definition is obtained.

(3) Quasivector spaces are called q-linear spaces in [7].

Clearly, every vector space is a quasivector one and every quasivector space is a quasilinear one. The following proposition is straightforward:

Proposition 1 (Markov [7]). *Let $(\mathfrak{M},+,\mathbb{R},*)$ be a quasilinear space over $\mathbb{R}$, and let $(\mathbb{Q},+)$, $\mathbb{Q}=\mathbb{D}(\mathfrak{M})$, be the induced abelian group. Let $*:\mathbb{R}\times\mathbb{Q}\to\mathbb{Q}$ be multiplication by scalars defined by (1). Then $(\mathbb{Q},+,\mathbb{R},*)$ is a quasivector space over $\mathbb{R}$.*

Conjugate elements. From $\mathrm{opp}(a)+a=0$ we obtain $\neg\mathrm{opp}(a)\neg a=0$, that is $\neg\mathrm{opp}(a)=\mathrm{opp}(\neg a)$. The element $\neg\mathrm{opp}(a)=\mathrm{opp}(\neg a)$ will be further denoted by a_- and the corresponding operator will be called *dualization* or *conjugation*.

Relations $\neg\mathrm{opp}(a)=\mathrm{opp}(\neg a)=a_-$ imply $\mathrm{opp}(a)=\neg(a_-)=(\neg a)_-$, which will be shortly denoted $\mathrm{opp}(a)=\neg a_-$. The last notation will be used to denote symbolically the opposite elements instead of the confusing notation $-a$ meaning opposite in algebra and negation in convex and interval analysis. Thus, in a quasivector space we write $a\neg a_-=0$, resp. $\neg a_-+a=0$.

Subspaces, direct sum. A *subspace of a quasivector space* $(\mathbb{Q},+,\mathbb{R},*)$ is a quasivector space $(\mathbb{P},+,\mathbb{R},*)$, such that $\mathbb{P}\subseteq\mathbb{Q}$ (the operations in $\mathbb{P}$ are inherited from $\mathbb{Q}$). If $(\mathbb{P},+,\mathbb{R},*)$ is a subspace of the quasivector space $(\mathbb{Q},+,\mathbb{R},*)$ then, of course, $(\mathbb{P},+)$ is an abelian subgroup of the abelian group $(\mathbb{Q},+)$. A sufficient condition for subspace can be formulated as follows. $\mathbb{H}$ is a subspace of the quasivector space $\mathbb{G}$ if and only if $\mathbb{H}\subset\mathbb{G}$ and $\mathbb{H}$ is closed under "$+$", "$*$", "$_-$", i.e.:

(i) $a+b\in\mathbb{H}$ for all $a,b\in\mathbb{H}$;
(ii) $\alpha*c\in\mathbb{H}$ for all $\alpha\in\mathbb{R}$ and $c\in\mathbb{H}$;
(iii) $a_-\in\mathbb{H}$ for all $a\in\mathbb{H}$.

Sum and direct sum of quasivector spaces are defined as in vector spaces. Namely, for two quasivector spaces U, V there is a least subspace containing both U and V, called their sum and written $U+V$. We have $U+V=\{u+v \mid u\in U, v\in V\}$. Let Z be a quasivector space and U,V be subspaces of Z. We say that Z is the *direct sum* of U and V and write $Z=U\oplus V$, if each $z\in Z$ can be uniquely presented in the form $z=u+v$, where $u\in U$, $v\in V$. One can show:

(1) a sum $U+V$ is direct, if $u_1+v_1=u_2+v_2$, $u_1,u_2\in U$, $v_1,v_2\in V$ imply $u_1=u_2$, $v_1=v_2$ (or, equivalently, $u+v=0$, $u\in U$, $v\in V$ imply $u=0$, $v=0$);
(2) $Z=U\oplus V \Leftrightarrow Z=U+V$ and $U\cap V=0$.

The elements of $U\oplus V$ are denoted $(u;v)$. Addition in $U\oplus V$ is $(u_1;v_1)+(u_2;v_2)=(u_1+u_2;v_1+v_2)$ and multiplication by scalars is $\gamma*(u;v)=(\gamma*u;\gamma*v)$.

2.3. Examples of quasivector spaces

Example 1. The system $(\mathfrak{K},+)$ of all convex bodies in a real m-dimensional Euclidean vector space $\mathbb{E}^m$ with set-theoretic (vector, Minkowski) addition: $A+B=\{\alpha+\beta \mid \alpha\in A, \beta\in B\}, A,B\in\mathfrak{K}$, is a proper abelian monoid with cancellation law having as a neutral element the origin "0" of $\mathbb{E}^m$ [17]. The system $(\mathfrak{K},+,\mathbb{R},*)$, where "$*$" is the set-theoretic multiplication by real scalars: $\gamma*A=\{\gamma\alpha \mid \alpha\in A\}$, is a quasilinear space (of monoid structure). The monoid $(\mathfrak{K},+)$ induces a group of generalized (extended, directed) convex bodies $(\mathbb{D}(\mathfrak{K}),+)$, which has been considered by a number of authors, cf. [1–3,11–14]. In [7] we investigate the space $(\mathbb{D}(\mathfrak{K}),+,\mathbb{R},*)$, where "$*$" is defined by (1). In particular n-dimensional intervals form a quasilinear space [5,6,10,15,16], which induces a quasivector space of generalized/directed intervals [8].

Example 2. For any integer $k\geqslant 1$ the set $\mathbb{R}^k$ of all k-tuples $(\alpha_1,\alpha_2,\ldots,\alpha_k)$, where $\alpha_i\in\mathbb{R}$ and $(\alpha_1,\alpha_2,\ldots,\alpha_k)$, $(\beta_1,\beta_2,\ldots,\beta_k)$ are distinct unless $\alpha_1=\beta_1,\alpha_2=\beta_2,\ldots,\alpha_k=\beta_k$, forms a quasivector space over $\mathbb{R}$ under the following operations:

$$(\alpha_1,\alpha_2,\ldots,\alpha_k)+(\beta_1,\beta_2,\ldots,\beta_k)=(\alpha_1+\beta_1,\alpha_2+\beta_2,\ldots,\alpha_k+\beta_k), \tag{8}$$

$$\gamma*(\alpha_1,\alpha_2,\ldots,\alpha_k)=(|\gamma|\alpha_1,|\gamma|\alpha_2,\ldots,|\gamma|\alpha_k), \quad \gamma\in\mathbb{R}. \tag{9}$$

This quasivector space will be denoted by $\mathbb{S}^k=(\mathbb{R}^k,+,\mathbb{R},*)$ and called the *canonical symmetric quasivector space* (of dimension k). Note that multiplication by -1 (negation) in $\mathbb{S}^k$ is the same as identity while the opposite operator is the same as conjugation:

$$\mathrm{opp}(\alpha_1,\alpha_2,\ldots,\alpha_k)=(\alpha_1,\alpha_2,\ldots,\alpha_k)_-=(-\alpha_1,-\alpha_2,\ldots,-\alpha_k). \tag{10}$$

Denoting $\mathbb{S}=\mathbb{S}^1$, we have $\mathbb{S}^k=\mathbb{S}\oplus\mathbb{S}\oplus\cdots\oplus\mathbb{S}$.

Remark. The case $k=1$ in the above Example 2 has been discussed to some extent in [15], see their Example 3.2.

Example 3. Consider the set of infinite sequences $(\alpha_1,\alpha_2,\ldots)$, $\alpha_i\in\mathbb{R}$, with addition and multiplication by scalars defined as in (8) and (9). We again obtain a quasivector space.

Example 4. The set of all real functions is a quasivector space if we define $f+g$ as the function whose value at x is $f(x)+g(x)$, and $\gamma * f$ as a function whose value at x is

$$\gamma * f(x) = \begin{cases} \gamma \cdot f(x) & \text{if } \gamma \geqslant 0, \\ |\gamma| \cdot f(-x) & \text{if } \gamma < 0. \end{cases} \tag{11}$$

In particular, negation is: $-1 * f(x) = f(-x)$. Note that in this quasivector space negation is distinct from opposite $\mathrm{opp}(f) = -f$. Note that the composition of opposite and negation $-f(-x)$ is a new operator. The operation (11) appears in the theory of (differences of) support functions, cf. [7,18]. We note that, if f is the support function of $A \in \mathfrak{K}$, then (11) is the support function of the convex body $\gamma * A$; in particular, $-1 * f(x) = f(-x)$ is the support function of $\neg A$.

Example 5. Let $\mathbb{C} = (\mathbb{C}, +, \mathbb{R}, \cdot)$ be the vector space of all complex numbers $c = c_1 + \mathrm{i}c_2$ with addition: $(c_1 + \mathrm{i}c_2) + (d_1 + \mathrm{i}d_2) = (c_1 + d_1) + \mathrm{i}(c_2 + d_2)$ and multiplication by real scalars: $\gamma \cdot (c_1 + \mathrm{i}c_2) = \gamma c_1 + \mathrm{i}\gamma c_2$. Opposite is $-c = -c_1 - \mathrm{i}c_2$. One introduces in $\mathbb{C}$ conjugate elements by means of: $c_- = \bar{c} = c_1 - \mathrm{i}c_2$; in particular $\bar{\mathrm{i}} = -\mathrm{i}$. Define a new multiplication by scalars in $\mathbb{C}$ by

$$\gamma * c = \begin{cases} \gamma \cdot c & \text{if } \gamma \geqslant 0, \\ \gamma \cdot \bar{c} & \text{if } \gamma < 0. \end{cases}$$

The system $\mathbb{C}^* = (\mathbb{C}, +, \mathbb{R}, *)$ is a quasivector space. Negation in $\mathbb{C}^*$ is $\neg c = (-1) * c = -\bar{c} = -(c_1 - \mathrm{i}c_2) = -c_1 + \mathrm{i}c_2$. We have $\mathbb{C}^* = \mathbb{V}^1 \oplus \mathbf{Im}$, where $\mathbf{Im} = (\mathbf{Im}, +, \mathbb{R}, *)$ is the quasivector space of purely imaginary numbers. Note that in **Im** negation is same as identity, whereas conjugation is same as opposite. Note that the "quasivector" multiplication "$*$" of a complex number c does not change the sign of the imaginary part c_2, whereas the "linear" multiplication "$\cdot$" may change it (whenever the scalar is negative).

Example 6. Consider the direct sum $\mathbb{V}^l \oplus \mathbb{S}^k$ of the l-dimensional vector space $\mathbb{V}^l = (\mathbb{R}^l, +, \mathbb{R}, \cdot)$ and the quasivector space $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$ from Example 2. The elements of $\mathbb{V}^l \oplus \mathbb{S}^k$ are n-tuples, $n = l + k$, of the form $(\lambda_1, \ldots, \lambda_l; \lambda_{l+1}, \ldots, \lambda_{l+k})$. Addition and multiplication by scalars ($\gamma \in \mathbb{R}$) are:

$$(\lambda_1, \ldots, \lambda_l; \lambda_{l+1}, \ldots, \lambda_{l+k}) + (\mu_1, \ldots, \mu_l; \mu_{l+1}, \ldots, \mu_{l+k})$$
$$= (\lambda_1 + \mu_1, \ldots, \lambda_l + \mu_l; \lambda_{l+1} + \mu_{l+1}, \ldots, \lambda_{l+k} + \mu_{l+k}),$$

$$\gamma * (\lambda_1, \ldots, \lambda_l; \lambda_{l+1}, \ldots, \lambda_{l+k}) = (\gamma\lambda_1, \ldots, \gamma\lambda_l; |\gamma|\lambda_{l+1}, \ldots, |\gamma|\lambda_{l+k}).$$

As direct sum of two quasivector spaces, $\mathbb{V}^l \oplus \mathbb{S}^k$ is a quasivector space. Negation

$$(-1) * (\lambda_1, \ldots, \lambda_l; \lambda_{l+1}, \ldots, \lambda_{l+k}) = (-\lambda_1, \ldots, -\lambda_l; \lambda_{l+1}, \ldots, \lambda_{l+k})$$

is distinct from opposite: $\mathrm{opp}(\lambda_1, \ldots, \lambda_{l+k}) = (-\lambda_1, \ldots, -\lambda_l; -\lambda_{l+1}, \ldots, -\lambda_{l+k})$. The composition of the opposite and negation operators yields:

$$\mathrm{opp}(\neg(\lambda_1, \ldots, \lambda_l;\ \lambda_{l+1}, \ldots, \lambda_{l+k})) = (\lambda_1, \ldots, \lambda_l; -\lambda_{l+1}, \ldots, -\lambda_{l+k}).$$

3. Calculation in quasivector spaces

3.1. Rules for calculation in quasivector spaces

Let $(\mathbb{Q}, +, \mathbb{R}, *)$ be a quasivector space over $\mathbb{R}$. As $(\mathbb{Q}, +)$ is a group, for every a there exists an opposite element $\mathrm{opp}(a) = \neg a_-$, such that $a \neg a_- = 0$. In a quasivector space the relation $a \neg a = 0$, may not necessarily hold; indeed, due to the condition $\alpha\beta \geqslant 0$ in (7) the equality $(-1) * a + 1 * a = (-1+1) * a$ may not be true. This means that $\neg a$ is not generally the opposite of a (unlike in a vector space, where negation and opposite coincide).

Using the group properties, such as $0 + a = a$, $\mathrm{opp}(a) + a = 0$, $\mathrm{opp}(a+b) = \mathrm{opp}(a) + \mathrm{opp}(b)$, $a + b = a + c \Rightarrow b = c$, and relations (4)–(7) one can derive rules for calculation in a quasivector space. A list of such rules is summarized in the following.

Proposition 2. *Let $(\mathbb{Q}, +, \mathbb{R}, *)$ be a quasivector space over $\mathbb{R}$. Then for all $\alpha, \beta, \gamma \in \mathbb{R}$ and all $a, b, c \in \mathbb{Q}$ the following properties hold:*

(1) $0 * a = 0$;
(2) $\gamma * 0 = 0$;
(3) $\mathrm{opp}(\gamma * a) = \gamma * \mathrm{opp}(a)$;
(4) $\neg(\gamma * a) = (-\gamma) * a$;
(5) $\gamma * (a \neg b) = \gamma * a \neg \gamma * b$;
(6) $\gamma * a = 0 \Rightarrow \gamma = 0$ *or* $a = 0$;
(7) $\gamma * a = \gamma * b \Rightarrow \gamma = 0$ *or* $a = b$;
(8) $(\alpha - \beta) * c = \alpha * c + (-\beta) * c = \alpha * c \neg \beta * c$, $\alpha\beta \leqslant 0$;
(9) $(\sum_{i=1}^{n} \alpha_i) * c = \sum_{i=1}^{n} \alpha_i * c, \alpha_i \geqslant 0,\ i = 1, \ldots, n$;
(10) $\alpha * \sum_{i=1}^{n} c_i = \sum_{i=1}^{n} \alpha * c_i$.

Proof. The verification of the above properties is trivial. For example, we prove the first three properties.

(1) We have $1 * a = (1+0) * a = 1 * a + 0 * a$, implying $0 * a = 0$.
(2) If $\gamma = 0$ the relation follows from 1); if $\gamma \neq 0$, then $c + \gamma * 0 = \gamma * ((1/\gamma) * c + 0) = \gamma * ((1/\gamma) * c) = c$, hence $\gamma * 0 = 0$.
(3) Assume $\gamma \neq 0$. We have to prove that $\gamma * \mathrm{opp}(a) + \gamma * a = 0$, that is, $\mathrm{opp}(a) + (1/\gamma) * (\gamma * a) = 0$, which is obviously true. □

Note that $y = a_-$ is the solution of the equation: $y \neg a = 0$, resp. $\neg y + a = 0$. For $\gamma \in \mathbb{R}$ and $a, b \in \mathbb{Q}$ we have the following relations using conjugation: $\gamma * (a \neg b_-) = \gamma * a \neg \gamma * b_-$; $a \neg a_- = \neg a + a_- = 0$; $\gamma * a \neg \gamma * a_- = \gamma * (a \neg a_-) = 0$; $a + b = 0 \Leftrightarrow a = \neg b_-$; $a + \gamma * b = 0 \Leftrightarrow a = (-\gamma) * b_- = \neg(\gamma * b_-)$.

We shall make use of the binary set $\Lambda = \{+, -\}$ and the function $\sigma : \mathbb{R} \to \Lambda$ defined by

$$\sigma(\gamma) = \begin{cases} + & \text{if } \gamma \geqslant 0, \\ - & \text{if } \gamma < 0. \end{cases}$$

The "product" $\lambda\mu$, $\lambda, \mu \in \Lambda$, is defined by $++ = -- = +$, $+- = -+ = -$.

A special symbolic notation. We make the convention $a_+ = a$. Then the symbolic notation a_λ for $a \in \mathbb{Q}$, $\lambda \in \Lambda$, makes sense; namely a_λ is either a or a_- according to the binary value of λ. Using this notation one may write rules holding true for all $a, b, c \in \mathbb{Q}, \alpha \in \mathbb{R}, \lambda, \mu, v \in \Lambda$, such as: $(a+b)_\lambda = a_\lambda + b_\lambda$; $(a_\mu + b_v)_\lambda = a_{\lambda\mu} + b_{\lambda v}$; $(\alpha * c_\mu)_v = \alpha * c_{\mu v}$, e.g., $(\alpha * c_\mu)_\mu = \alpha * c$. The possibility to perform such symbolic transformations justifies the use of the notation a_- for conjugate instead of the traditional notation $\bar{a}$. More motivations in this direction are contained in the sequel.

3.2. The quasidistributive law

The condition $\alpha\beta \geqslant 0$ in (7) makes the impression that there may be some freedom in the form of the distributivity relation for $\alpha\beta < 0$. The following two theorems show that this is not the case: it turns out that (7) determines a special relation for all $\alpha, \beta \in \mathbb{R}$.

Theorem 1. *Let $(\mathbb{Q}, +, \mathbb{R}, *)$ be a quasivector space over $\mathbb{R}$. For $\alpha, \beta \in \mathbb{R}$ and $c \in \mathbb{Q}$ we have:*

$$(\alpha + \beta) * c_{\sigma(\alpha+\beta)} = \alpha * c_{\sigma(\alpha)} + \beta * c_{\sigma(\beta)}. \tag{12}$$

Proof. In the case $\sigma(\alpha) = \sigma(\beta)$ (12) is true by assumption (7). Consider the case $\sigma(\alpha) = -\sigma(\beta)$. Assume that $0 \leqslant \alpha$, $\beta < 0$ and $0 < -\beta \leqslant \alpha$. In this subcase we have $0 \leqslant \alpha + \beta$, so that (12) reads: $(\alpha + \beta) * c = a * c + \beta * c_-$. Using (7), we can write

$$\begin{aligned} \alpha * c + \beta * c_- &= ((\alpha + \beta) - \beta) * c + \beta * c_- \\ &= (\alpha + \beta) * c \neg \beta * c + \beta * c_- \\ &= (\alpha + \beta) * c + \beta * (\neg c + c_-) = (\alpha + \beta) * c, \end{aligned}$$

so that (12) is proved to hold true in this subcase. The remaining subcases are verified similarly. □

"Dualizing" by $\sigma(\alpha + \beta)$, relation (12) can be written in the equivalent form

$$(\alpha + \beta) * c = \alpha * c_{\sigma(\alpha)\sigma(\alpha+\beta)} + \beta * c_{\sigma(\beta)\sigma(\alpha+\beta)}. \tag{13}$$

Relations (12) and (13) are convenient for symbolic computations. By contrast, without the use of binary variables, formula (13) obtains the form

$$(\alpha + \beta) * c = \begin{cases} \alpha * c + \beta * c & \text{if } \alpha\beta \geqslant 0, \\ \alpha * c + \beta * c_- & \text{if } \alpha\beta < 0, |\alpha| \geqslant |\beta|, \\ \alpha * c_- + \beta * c & \text{if } \alpha\beta < 0, |\alpha| < |\beta|, \end{cases}$$

which can be hardly used for symbolic manipulations.

Relation (12) shows that c can be always factored out in an expression of the form $\alpha * c_{\sigma(\alpha)} + \beta * c_{\sigma(\beta)}$. It is easy to verify that c cannot be generally factored out in an expression of the form $\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)}$. In fact, the following holds:

Theorem 2. *Let* $(\mathbb{Q}, +, \mathbb{R}, *)$ *be a quasivector space over* $\mathbb{R}$. *For* $\alpha, \beta \in \mathbb{R}$ and $c \in \mathbb{Q}$ *we have:*

$$\alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} = \begin{cases} \sigma(\alpha)|\alpha - \beta| * c_{\sigma(\alpha-\beta)} + \varepsilon * (c + c_-) & \text{if } \alpha\beta \geqslant 0, \\ (\alpha + \beta) * c_{\sigma(\alpha)} + \varepsilon * (c \neg c)_{\sigma(\alpha)} & \text{if } \alpha\beta < 0, \end{cases}$$

wherein

$$\varepsilon = \begin{cases} \alpha & \text{if } |\alpha| \leqslant |\beta|, \\ \beta & \text{if } |\alpha| > |\beta|. \end{cases}$$

Proof. Assume $\alpha\beta \geqslant 0$, $0 \leqslant \beta < \alpha$. We have

$$\begin{aligned} \alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} &= ((\alpha - \beta) + \beta) * c_{\sigma(\alpha-\beta)} + \beta * c_{-\sigma(\beta)} \\ &= (\alpha - \beta) * c_{\sigma(\alpha-\beta)} + \beta * c + \beta * c_- \\ &= (\alpha - \beta) * c_{\sigma(\alpha-\beta)} + \beta * (c + c_-). \end{aligned}$$

The remaining subcases of $\alpha\beta \geqslant 0$ are treated similarly. Assume now $\alpha\beta < 0$, $0 \leqslant -\beta < \alpha$. We have

$$\begin{aligned} \alpha * c_{\sigma(\alpha)} + \beta * c_{-\sigma(\beta)} &= ((\alpha + \beta) - \beta) * c_{\sigma(\alpha)} + \beta * c \\ &= (\alpha + \beta) * c_{\sigma(\alpha)} \neg \beta * c + \beta * c \\ &= (\alpha + \beta) * c_{\sigma(\alpha)} + \beta * (c \neg c)_{\sigma(\alpha)}. \end{aligned}$$

The remaining subcases are verified similarly. □

Relations (12) and (13) show that in a quasivector space one can "open brackets" in expressions of the form $(\alpha + \beta) * c$. However, it is not always possible to "factor out" a common multiplier c in an expression of the form $\alpha * c + \beta * c$. Indeed, it follows from Theorems 1 and 2 that

$$\alpha * c + \beta * c = (\alpha + \beta) * c + \gamma * (c \neg c), \tag{14}$$

wherein

$$\gamma = \begin{cases} 0 & \text{if } \alpha\beta \geqslant 0, \\ \alpha & \text{if } \alpha\beta < 0, |\alpha| \leqslant |\beta|, \\ \beta & \text{if } \alpha\beta < 0, |\alpha| > |\beta|. \end{cases}$$

We note that the rule $\alpha * c = \beta * c \Rightarrow \alpha = \beta$ or $c = 0$ is not generally valid in a quasivector space, rather we have $\alpha * c = \beta * c \Rightarrow \alpha = \beta$ or $c = 0$ or $c = \neg c, \alpha = -\beta$, which can be proved using (12).

4. Quasivector spaces and associated vector spaces

4.1. Vector spaces induced by quasivector spaces

Let $(\mathbb{Q}, +, \mathbb{R}, *)$ be a quasivector space over the l.o. field $\mathbb{R}$. Consider the operation "$\cdot$": $\mathbb{R} \times \mathbb{Q} \to \mathbb{Q}$ defined by

$$\alpha \cdot c = \alpha * c_{\sigma(\alpha)}, \quad \alpha \in \mathbb{R}, \;\; c \in \mathbb{Q}. \tag{15}$$

It is not difficult to prove [7]:

Theorem 3. *Let* $(\mathbb{Q}, +, \mathbb{R}, *)$ *be a quasivector space over* $\mathbb{R}$. *Then* $(\mathbb{Q}, +, \mathbb{R}, \cdot)$, *with* "$\cdot$" *defined by* (15), *is a vector space over* $\mathbb{R}$, *that is*

$$\alpha \cdot (\beta \cdot c) = (\alpha\beta) \cdot c, \tag{16}$$

$$\gamma \cdot (a + b) = \gamma \cdot a + \gamma \cdot b, \tag{17}$$

$$1 \cdot a = a, \tag{18}$$

$$(\alpha + \beta) \cdot c = \alpha \cdot c + \beta \cdot c. \tag{19}$$

Operation (15) is well defined on $\mathbb{R} \times \mathbb{Q}$ for any quasivector space $\mathbb{Q}$ over $\mathbb{R}$. This operation is called *linear multiplication* in $\mathbb{Q}$. By contrast, the original multiplication "$*$" in $\mathbb{R} \times \mathbb{Q}$ is called *quasivector multiplication*. Theorem 3 implies that every quasivector space $(\mathbb{Q}, +, \mathbb{R}, *)$ involves a linear multiplication and hence an *associated vector space* $(\mathbb{Q}, +, \mathbb{R}, \cdot)$. Note that the element $(-1) \cdot a = (-1) * a_- = \neg a_-$ is the opposite to a in $\mathbb{Q}$, $\mathrm{opp}(a) = \neg a_-$, as for $a \in \mathbb{Q}$ we have: $a + (-1) \cdot a = 0$.

Let $(\mathbb{Q}, +, \mathbb{R}, *)$ be a quasivector space over $\mathbb{R}$ and let $(\mathbb{Q}, +, \mathbb{R}, \cdot)$ be the induced via (15) vector space over $\mathbb{R}$. Consider the system $(\mathbb{Q}, +, \mathbb{R}, \cdot, -)$, which is the vector space $(\mathbb{Q}, +, \mathbb{R}, \cdot)$ together with the operator conjugation from the original space. Although the operation "$*$" is not explicitly present in the notation $(\mathbb{Q}, +, \mathbb{R}, \cdot, -)$, the latter system implicitly involves the operation "$*$", since using (15) we have

$$\alpha * c = \alpha \cdot c_{\sigma(\alpha)} = \begin{cases} \alpha \cdot c & \text{if } \alpha \geqslant 0, \\ \alpha \cdot c_- & \text{if } \alpha < 0. \end{cases} \tag{20}$$

Definition 3. Two algebraic systems with the same sets of elements and different sets of operations are *equivalent* if every expression in the first system can be presented in terms of the operations of the second system and vice versa.

Given a quasivector space $(\mathbb{Q}, +, \mathbb{R}, *)$ with conjugation a_-, the *extended vector space* $(\mathbb{Q}, +, \mathbb{R}, \cdot, _-)$ with "$\cdot$" defined by (15) is equivalent to the original quasivector space $(\mathbb{Q}, +, \mathbb{R}, *)$. The equivalence follows from the transition formulae (15), (20) and the relation $a_- = \neg\mathrm{opp}(a)$. Note that the vector space $(\mathbb{Q}, +, \mathbb{R}, \cdot)$ (without conjugation!), is generally not equivalent to the quasivector space $(\mathbb{Q}, +, \mathbb{R}, *)$.

The linear multiplication is useful for the interpretation of properties of the quasivector spaces in terms of known linear concepts. For example [7], let $\alpha, \beta \in \mathbb{R}$, $d \in \mathbb{Q}$. If $\gamma = \alpha^2 - \beta^2 \neq 0$, then the equation $\alpha \cdot x + \beta \cdot x_- = d$ has a unique solution $x = \gamma^{-1} \cdot (\alpha \cdot d - \beta \cdot d_-)$. If $\beta = \alpha \neq 0$, then $x + x_- = \alpha^{-1} \cdot d$. If $\beta = -\alpha \neq 0$, then $x - x_- = \alpha^{-1} \cdot d$. (Note that in the context of Example 5 the equation $\alpha \cdot x + \beta \cdot x_- = d$ can be written $\alpha x + \beta\bar{x} = d$.)

In the special case of convex bodies the linear multiplication has been introduced by means of (3) cf. [3,12,14]. Note that the right-hand sides of (3) and (15) are equivalent, in the sense that

$$\gamma * (A,B)_{\sigma(\gamma)} = \begin{cases} (\gamma * A, \gamma * B) & \text{if } \gamma \geqslant 0, \\ ((-\gamma) * B, (-\gamma) * A) & \text{if } \gamma < 0. \end{cases} \tag{21}$$

4.2. Quasivector spaces induced by vector spaces

Vector spaces with involution. Let $\mathbb{G} = (\mathbb{G}, +, \mathbb{R}, \cdot)$ be a vector space over the l.o. field $\mathbb{R}$. A linear transformation $i : \mathbb{G} \to \mathbb{G}$, such that for $a, b \in \mathbb{G}$, $\alpha \in \mathbb{R}$:

(1) $i(a + b) = i(a) + i(b)$,
(2) $i(\alpha \cdot c) = \alpha \cdot i(c)$, satisfying the additional assumption:
(3) $i^2(a) = a$,

is called an *involution* (*dual automorphism*, *symmetry*) in $\mathbb{G}$.

Note that an involution also satisfies: $i(a) = 0$ iff $a = 0$. In every vector space $(\mathbb{G}, +, \mathbb{R}, \cdot)$, there are (at least) two involutions: identity and opposite.

Assume that $\mathbb{G}$ is a vector space and i is an involution in $\mathbb{G}$. Define $* : \mathbb{R} \times \mathbb{G} \to \mathbb{G}$ by

$$\alpha * c = \begin{cases} \alpha \cdot c & \text{if } \alpha \geqslant 0, \\ \alpha \cdot i(c) & \text{if } \alpha < 0. \end{cases} \tag{22}$$

The next result shows that a vector space with involution generates a quasivector space; this result in some sense inverses Theorem 3:

Theorem 4. *Let $\mathbb{G} = (\mathbb{G}, +, \mathbb{R}, \cdot, i)$ be a vector space (over the l.o. field $\mathbb{R}$) with an involution i. Let $* : \mathbb{R} \times \mathbb{G} \to \mathbb{G}$ be defined by (22). Then:*

(1) *the system $(\mathbb{G}, +, \mathbb{R}, *)$ induced by $(\mathbb{G}, +, \mathbb{R}, \cdot, i)$ is a quasivector space;*
(2) *the two systems $(\mathbb{G}, +, \mathbb{R}, *)$ and $(\mathbb{G}, +, \mathbb{R}, \cdot, i)$ are equivalent.*

Proof.

(1) We are given that $(\mathbb{G}, +)$ is an abelian group and "$\cdot$" satisfies relations (16)–(19). We prove that relations (4)–(7) hold true with "$*$" defined by (22). To prove $\gamma * (a + b) = \gamma * a + \gamma * b$ assume first $\gamma \geqslant 0$. Then, using (22) and (17) we obtain (4). From (17) and the properties of i we have $\gamma \cdot i(a + b) = \gamma \cdot i(a) + \gamma \cdot i(b)$. Using (22) for $\gamma < 0$ the latter implies (4). To prove (5) we consider various cases. Let $\alpha \geqslant 0$, $\beta \geqslant 0$. In this case (5) holds because $\alpha * c = \alpha \cdot c$ for all $c \in \mathbb{G}$. Let $\alpha \geqslant 0$, $\beta < 0$. In this case we have $\beta * c = \beta \cdot i(c)$ for all $c \in \mathbb{G}$, resp., using the properties of i, we obtain $\beta \cdot c = \beta * i(c)$ and $(\alpha\beta) \cdot c = (\alpha\beta) * i(c)$. Replacing in (16) we obtain $\alpha * (\beta * i(c)) = (\alpha\beta) * i(c)$, which implies (5). The case $\alpha < 0$, $\beta \geqslant 0$ is proved analogously, the case $\alpha < 0$, $\beta < 0$ requires the property $i^2(a) = a$. Properties (4), (5) are proved, properties (6)–(7) can be proved similarly.

(2) To prove equivalence consider the involution i as conjugation in $(\mathbb{G},+,\mathbb{R},*)$, that is $i(a)=a_-$. □

The involution i generating a quasivector space in the sense of Theorem 4 will be called a *generic* involution. If we chose identity as a generic involution, then (22) implies $\alpha * c=\alpha\cdot c$ for all $\alpha\in\mathbb{R}$, $c\in\mathbb{G}$, that is both multiplications "·" and "∗" coincide; this is a trivial situation, as the induced quasivector space coincides with the original vector space. If opposite is taken as generic involution, then (22) obtains the form:

$$\alpha * c=|\alpha|\cdot c. \tag{23}$$

According to Theorem 4 the induced space $(\mathbb{G},+,\mathbb{R},*)$ with "∗" defined by (23) is a quasivector space. It is easy to check that the space $(\mathbb{G},+,\mathbb{R},*)$ is not linear in general. From (23) we obtain for $\alpha=-1$ that $(-1)*c=c$. We pay a special attention to this important case in Section 4.

The elements $c\in\mathbb{G}=(\mathbb{G},+,\mathbb{R},*)$, satisfying $c+i(c)=0$ form a subspace of $\mathbb{G}$; the elements $c\in\mathbb{G}$, such that $i(c)=c$, form another subspace. These elements play the role of symmetric, resp. linear elements to be considered next.

4.3. Representation of a quasivector space as a direct sum of a vector and a symmetric subspace

Definition 4. Q is a quasivector space. An element $a\in\mathbb{Q}$ with $a\neg a=0$ is called *linear*. An element $a\in\mathbb{Q}$ with $\neg a=a$ is called *symmetric*.

It can be easily checked that in a quasivector space $\mathbb{Q}$ the subsets of linear and symmetric elements $\mathbb{Q}'=\{a\in\mathbb{Q}\mid a\neg a=0\}$, resp. $\mathbb{Q}''=\{a\in\mathbb{Q}\mid a=\neg a\}$ form subspaces of $\mathbb{Q}$.

Proposition 3. *Assume that Q is a quasivector space. The subspace $\mathbb{Q}'=\{a\in\mathbb{Q}\mid a\neg a=0\}$ is a vector space.*

Indeed, we only have to check that relation (7) becomes true for all values of the scalars. However, this is obvious from (14).

Definition 5. Assume that Q is a quasivector space. The space $\mathbb{Q}'=\{a\in\mathbb{Q}\mid a\neg a=0\}$ is called the *linear subspace* of $\mathbb{Q}$ and the space $\mathbb{Q}''=\{a\in\mathbb{Q}\mid a=\neg a\}$ is called the *symmetric subspace* of $\mathbb{Q}$.

Below we summarize some of the properties of the linear and symmetric elements:

(1) $a\in\mathbb{Q}'\Leftrightarrow a=a_-\Leftrightarrow a\neg a=0\Leftrightarrow \neg a=\mathrm{opp}(a)\Leftrightarrow \exists c\in\mathbb{Q}: a=c+c_-$;
(2) $b\in\mathbb{Q}''\Leftrightarrow b=\neg b\Leftrightarrow b+b_-=0\Leftrightarrow b_-=\mathrm{opp}(b)\Leftrightarrow \exists d\in\mathbb{Q}: b=d\neg d$.

To prove existence, in case 1 take $c=(1/2)*a+s$, where $s\in\mathbb{Q}''$ is arbitrary, and in case 2 take $d=(1/2)*b+t$, where $t\in\mathbb{Q}'$ is arbitrary.

The next theorem shows that every quasivector space is a direct sum of a vector space and a symmetric quasivector space.

Theorem 5. *For every quasivector space* $\mathbb{Q}$ *we have* $\mathbb{Q} = \mathbb{Q}' \oplus \mathbb{Q}''$. *More specifically, for every* $x \in \mathbb{Q}$ *we have* $x = x' + x''$ *with unique*

$$x' = (1/2) * (x + x_-) \in \mathbb{Q}',$$

$$x'' = (1/2) * (x \neg x) \in \mathbb{Q}''.$$

Proof. Assume $x \in \mathbb{Q}$. Using that $x_- \neg x = 0$ we have

$$x' + x'' = (1/2) * (x + x_-) + (1/2) * (x \neg x)$$

$$= (1/2) * (x + x + x_- \neg x) = x.$$

On the other side we have $x' = (1/2) * (x + x_-) \in \mathbb{Q}'$ and $x'' = (1/2) * (x \neg x) \in \mathbb{Q}''$. Hence, $\mathbb{Q} = \mathbb{Q}' + \mathbb{Q}''$. Furthermore, $\mathbb{Q}' \cap \mathbb{Q}'' = 0$. Indeed, assume $x \in \mathbb{Q}'$ and $x \in \mathbb{Q}''$. Then we have simultaneously $x \neg x = 0$ and $x = \neg x$, implying $x = 0$. Hence $\mathbb{Q} = \mathbb{Q}' \oplus \mathbb{Q}''$. □

Theorem 5 states that every element $x \in \mathbb{Q}$ can be decomposed in a unique way as $x' + x''$, where x' is an element of a vector space and x'' belongs to a symmetric quasivector space. We shall call x' the linear part of x, and x''—the symmetric part of x, and write $x = (x'; x'')$.

In a quasivector space $\mathbb{Q}$ the symmetry property $a = \mathrm{opp}(a)$ is satisfied only by the null element 0 of $\mathbb{Q}$. Indeed, $a = \neg a_-$ is equivalent to $a + a = 0$, or $2 * a = 0$, resp. $a = 0$.

Since the distributivity relation is of different form in a vector, resp. symmetric quasivector space, one may wonder how this fact agrees with Theorem 5. Indeed, assume $c \in \mathbb{Q}$ with $c = c' + c''$, $c' \in \mathbb{Q}'$, $c'' \in \mathbb{Q}''$, equivalently: $c = (c'; c'')$. Relation (12) decomposes into

$$(\alpha + \beta) * c'_{\sigma(\alpha+\beta)} = \alpha * c'_{\sigma(\alpha)} + \beta * c'_{\sigma(\beta)},$$

$$(\alpha + \beta) * c''_{\sigma(\alpha+\beta)} = \alpha * c''_{\sigma(\alpha)} + \beta * c''_{\sigma(\beta)}.$$

Using that c' is linear, we have $c' = c'_-$, so that the first relation is equivalent to the familiar: $(\alpha + \beta) * c' = \alpha * c' + \beta * c'$.

Hints for practical applications. In practice we need to know how to solve problems formulated in quasilinear spaces with monoid structures, like convex bodies and intervals, cf. Example 1 (Section 2). Assume that $\mathfrak{M}$ is a quasilinear space (with monoid structure), cf. Definition 1, and $\mathbb{Q} = \mathbb{D}(\mathfrak{M})$ is the induced quasivector space of factorized pairs (A, B), $A, B \in \mathfrak{M}$. Let us check how proper elements of the form $a = (A, 0)$ are decomposed in the form $a = a' + a'' = (a'; a'')$. We have

$$a' = (1/2) * (a + a_-) = (1/2) * (A, \neg A), \tag{24}$$

$$a'' = (1/2) * (a \neg a) = (1/2) * (A \neg A, 0). \tag{25}$$

We see that the linear part $a' = (A, \neg A)$ of a proper element $(A, 0)$ may not be a proper element in general. The linear part $(A, \neg A)$ is a proper element if there exists $X \in \mathfrak{M}$ such that $(A, \neg A) = (X, 0)$, that is $A = X \neg A$. For example, in the case of two-dimensional convex bodies, if A is a (proper) triangle, then such X does not exist and consequently the linear part of A is an improper element. In such a situation we need to interpret results which are improper in terms of proper elements.

5. Symmetric quasivector spaces

5.1. Relation to vector spaces

Recall that an element s from a quasivector space, such that $(-1)*s=s$, briefly $\neg s=s$, is called (*centrally*) *symmetric*. A quasivector space consisting of symmetric elements is called a *symmetric quasivector space*.

A symmetric quasivector space $\mathfrak{S}$ can be defined axiomatically as an abelian group with multiplication by scalars (from a l.o. field) satisfying (4)–(7) together with the additional assumption: $\neg a=a$ for all $a\in\mathfrak{S}$.

In a symmetric quasivector space we have: $\alpha * c=(-\alpha)*c=|\alpha|*c$. Hence formula (15) for the induced linear multiplication in a symmetric quasivector space can be written as

$$\alpha\cdot c=|\alpha|*c_{\sigma(\alpha)}. \tag{26}$$

Relation (23) shows that in a vector space one can introduce a quasivector multiplication by scalars via $\alpha * c=|\alpha|\cdot c$, thus using the available operations in a vector space over the l.o. field $\mathbb{R}$. Substituting $\alpha=-1$ in (23), we obtain $\neg c=1\cdot c=c$. This shows that negation coincides with identity (all elements c are symmetric); hence negation can be expressed by means of the operations in the vector space. Also, recall that for symmetric elements a conjugation equals opposite,

$$a_-=\mathrm{opp}(\neg a)=\mathrm{opp}(a). \tag{27}$$

We thus obtain the following corollary from Theorem 3:

Corollary. *Let* $(\mathfrak{S},+,\mathbb{R},*)$ *be a symmetric quasivector space over* $\mathbb{R}$. *The induced vector space* $(\mathfrak{S},+,\mathbb{R},\cdot)$, *with* "$\cdot$" *defined by* (15), *due to* (23), (27), *has all operations present in the symmetric quasivector space* $(\mathfrak{S},+,\mathbb{R},*)$, *and thus* $(\mathfrak{S},+,\mathbb{R},*)$ *and* $(\mathfrak{S},+,\mathbb{R},\cdot)$ *are equivalent.*

Conversely, we have

Theorem 6. *Every vector space over a l.o. field* $\mathbb{R}$ *induces via* (23), (27) *an equivalent symmetric quasivector space.*

Thus to every vector space over a l.o. field $(\mathbb{G},+,\mathbb{R},\cdot)$, we associate the symmetric quasivector space $(\mathbb{G},+,\mathbb{R},*)$ with "$*$" defined by (23). In accordance to what was said above we distinguish between the two multiplications by scalars: the *linear multiplication by scalars* "$\cdot$", and the *quasivector* multiplication by scalars "$*$". Note that the two spaces—the original $(\mathbb{Q},+,\mathbb{R},*)$ and the induced $(\mathbb{Q},+,\mathbb{R},\cdot)$—although equivalent are generally distinct from each other as they generally have different operations for multiplication by scalars; the quasivector multiplication in a symmetric quasivector space is generally not a linear multiplication.

A vector space can be endowed with several involutions; when used as generic involutions, the latter may generate distinct quasivector spaces. In particular, if a vector space has a generic involution i, distinct from identity and opposite, then the generated quasivector space via (22) is generally distinct from the symmetric quasivector space (induced via the opposite operator).

Setting $\alpha=-1$ in (22) we obtain $(-1)*c=(-1)\cdot i(c)=\mathrm{opp}(i(c))$. Thus i coincides with conjugation: $i(c)=\mathrm{opp}(\neg c)=c_-$. Hence formula (22) becomes of form (20), which completes the relation between a quasivector space $(\mathbb{Q}, +, \mathbb{R}, *)$ and the induced vector space $(\mathbb{Q}, +, \mathbb{R}, \cdot)$. More specifically, every quasivector space $(\mathbb{Q}, +, \mathbb{R}, *)$ is equivalent to the induced vector space endowed with the corresponding conjugation operation, that is the spaces $(\mathbb{Q}, +, \mathbb{R}, *)$ and $(\mathbb{Q}, +, \mathbb{R}, \cdot, \_)$ are equivalent.

According to Theorem 4 every quasivector space is equivalent to a vector space endowed with an involution i (called conjugation). In the special case, when this involution coincides with some of the involutions in the associated vector space (identity or opposite), then the quasivector space is either linear or symmetric. If i is neither identity nor opposite, then the quasivector space has (at least) four involutions: identity, opposite, conjugation (i) and negation opp(i), which is a composition of opposite and conjugation.

The "symmetric" case can be summarized as follows: *Every symmetric quasivector space over* $\mathbb{R}$ *generates via* (26) *an equivalent vector space and*, *vice versa*, *every vector space over* $\mathbb{R}$ *induces via* (20) *an equivalent symmetric quasivector space*.

5.2. Linear combinations in symmetric quasivector spaces

Assume that $(\mathfrak{S}, +, \mathbb{R}, *)$ is a *symmetric* quasivector space and $(\mathfrak{S}, +, \mathbb{R}, \cdot)$ is the associated equivalent vector space. From the vector space $(\mathfrak{S}, +, \mathbb{R}, \cdot)$ we may transfer vector space concepts, such as linear combination, linear dependence, basis etc., to the original symmetric quasivector space $(\mathfrak{S}, +, \mathbb{R}, *)$. For example, the concept of linear combination obtains the following form.

Let $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$ be finitely many (not necessarily distinct) elements of $\mathfrak{S}$. An element $f \in \mathfrak{S}$ of the form

$$f = \alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(2)}_{\sigma(\alpha_2)} + \cdots + \alpha_k * c^{(k)}_{\sigma(\alpha_k)}, \tag{28}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$, is called a *linear combination* of $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathfrak{S}$.

Remarks. (1) Using (15) we see that (28) is a reformulation of the familiar linear combination $f = \sum_{i=1}^{k} \alpha_i \cdot c^{(i)} = \alpha_1 \cdot c^{(1)} + \alpha_2 \cdot c^{(2)} + \cdots + \alpha_k \cdot c^{(k)}$ from the induced vector space $(\mathfrak{S}, +, \mathbb{R}, \cdot)$. (2) It is shown in [7] that the concept of linear combination can be directly extended to an arbitrary (not necessarily symmetric) quasivector space; here we use a more restricted, but simpler approach.

Proposition 4. *Let* $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathfrak{S}$, $k \geqslant 1$. *Then the set*

$$\mathbb{H} = \left\{ \sum_{i=1}^{k} \alpha_i * c^{(i)}_{\sigma(\alpha_i)} \mid \alpha_i \in \mathbb{R}, i = 1, \ldots, k \right\}$$

of all linear combinations of $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$ *is a subspace of* $\mathfrak{S}$.

The proof is elementary—it can be done either by passing to the induced vector space or in terms of quasivector spaces. Throughout this section we shall keep to the second approach.

The elements $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$ form a *generating set* for $\mathbb{H}$. We also say that the subspace $\mathbb{H}$ defined in Proposition 4 is *spanned* by $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$ and write $\mathbb{H} = \text{span}\ \{c^{(1)}, c^{(2)}, \ldots, c^{(k)}\}$.

Let $\mathfrak{S}$ be a symmetric quasivector space over $\mathbb{R}$. The elements $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathfrak{S}$, $k \geqslant 1$, are *linearly dependent* (*over* $\mathbb{R}$), if there exists a nontrivial linear combination of $\{c^{(i)}\}$, which is equal to 0, i.e. if there exists a system $\{\alpha_i\}_{i=1}^k$ with not all α_i equal to zero, such that

$$\alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(1)}_{\sigma(\alpha_2)} + \cdots + \alpha_k * c^{(k)}_{\sigma(\alpha_k)} = 0. \tag{29}$$

Elements of $\mathfrak{S}$, which are not linearly dependent, are *linearly independent*. That is, the elements $c^{(1)}, c^{(2)}, \ldots, c^{(k)} \in \mathfrak{S}$ are *linearly independent*, if (29) is possible only for the trivial linear combination, such that $\alpha_i = 0$ for all $i = 1, \ldots, k$.

5.3. Linear mappings in quasivector spaces

Let $\mathbb{Q}_1 = (\mathbb{Q}_1, +, \mathbb{R}, *)$, $\mathbb{Q}_2 = (\mathbb{Q}_2, +, \mathbb{R}, *)$ be two quasivector spaces over $\mathbb{R}$ and let $\varphi : \mathbb{Q}_1 \to \mathbb{Q}_2$ be a homomorphic (linear) mapping, that is

$$\varphi(x + y) = \varphi(x) + \varphi(y), \tag{30}$$

$$\varphi(\lambda * x) = \lambda * \varphi(x), \quad x, y \in \mathbb{Q}_1, \ \lambda \in \mathbb{R}. \tag{31}$$

It is easy to check that $\varphi(x_-) = (\varphi(x))_-$; more generally any linear mapping satisfies:

$$\begin{aligned}&\varphi(\alpha_1 * x^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * x^{(2)}_{\sigma(\alpha_2)} + \cdots + \alpha_k * x^{(k)}_{\sigma(\alpha_k)})\\ &\quad = \alpha_1 * \varphi(x^{(1)})_{\sigma(\alpha_1)} + \alpha_2 * \varphi(x^{(2)})_{\sigma(\alpha_2)} + \cdots + \alpha_k * \varphi(x^{(k)})_{\sigma(\alpha_k)},\end{aligned} \tag{32}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$, $x^{(1)}, x^{(2)}, \ldots, x^{(k)} \in \mathbb{Q}_1$. In particular:

$$\varphi(\alpha * x_\lambda + \beta * y_\mu) = \alpha * \varphi(x)_\lambda + \beta * \varphi(y)_\mu, \quad x, y \in \mathbb{Q}_1, \ \lambda, \mu \in \mathbb{R}. \tag{33}$$

Obviously condition (33) completely characterizes a linear mapping and can substitute conditions (30) and (31).

Let $\mathfrak{S}$ be a symmetric quasivector space and $x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in \mathfrak{S}$ and let $\mathbb{S}^n = (\mathbb{R}^n, +, \mathbb{R}, *)$ be the canonic symmetric quasivector space defined in Example 2. The mapping $\varphi : \mathbb{S}^n \to \mathfrak{S}$, such that

$$\varphi(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha_1 * x^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * x^{(2)}_{\sigma(\alpha_2)} + \cdots + \alpha_n * x^{(n)}_{\sigma(\alpha_n)}, \tag{34}$$

is linear. Indeed, using Theorem 1 we have

$$\begin{aligned}\varphi((\alpha_1, \alpha_2, \ldots, \alpha_n) + (\beta_1, \beta_2, \ldots, \beta_n)) &= \varphi(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots, \alpha_n + \beta_n)\\ &= (\alpha_1 + \beta_1) * x^{(1)}_{\sigma(\alpha_1+\beta_1)} + \cdots + (\alpha_n + \beta_n) * x^{(n)}_{\sigma(\alpha_n+\beta_n)}\\ &= \alpha_1 * x^{(1)}_{\sigma(\alpha_1)} + \cdots + \alpha_n * x^{(n)}_{\sigma(\alpha_n)} + \beta_1 * x^{(1)}_{\sigma(\beta_1)} + \cdots + \beta_n * x^{(n)}_{\sigma(\beta_n)}\\ &= \varphi(\alpha_1, \alpha_2, \ldots, \alpha_n) + \varphi(\beta_1, \beta_2, \ldots, \beta_n);\end{aligned}$$

$$\begin{aligned}\varphi(\gamma * (\alpha_1, \alpha_2, \ldots, \alpha_n)) &= \varphi(|\gamma|\alpha_1, |\gamma|\alpha_2, \ldots, |\gamma|\alpha_n)\\ &= (|\gamma|\alpha_1) * x^{(1)}_{\sigma(|\gamma|\alpha_1)} + \cdots + (|\gamma|\alpha_n) * x^{(k)}_{\sigma(|\gamma|\alpha_n)})\\ &= (|\gamma|\alpha_1) * x^{(1)}_{\sigma(\alpha_1)} + \cdots + (|\gamma|\alpha_n) * x^{(k)}_{\sigma(\alpha_n)})\\ &= |\gamma| * \varphi(\alpha_1, \alpha_2, \ldots, \alpha_n) = \gamma * \varphi(\alpha_1, \alpha_2, \ldots, \alpha_n).\end{aligned}$$

Denote $e^{(i)} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$, where the component 1 is on the i-th place. We consider $e^{(i)}$ as elements of $\mathbb{S}^n$, where $\mathrm{opp}(e^{(i)}) = e_{-}^{(i)}$ and $\neg e^{(i)} = e^{(i)}$. Relation (34) implies

$$\varphi(e^{(i)}) = \alpha_i * x^{(i)}_{\sigma(\alpha_i)}|_{\alpha_i=1} = x^{(i)}, \quad i = 1, \ldots, n. \tag{35}$$

The mapping φ is the only linear mapping from $\mathbb{S}^n$ to $\mathfrak{S}$ with the property (35). Indeed, if (35) holds, then by (32),

$$\varphi(\alpha_1, \alpha_2, \ldots, \alpha_n) = \varphi\left(\sum \alpha_i * e^{(i)}_{\sigma(\alpha_i)}\right)$$

$$= \sum \alpha_i * \varphi(e^{(i)})_{\sigma(\alpha_i)} = \sum \alpha_i * x^{(i)}_{\sigma(\alpha_i)}.$$

We thus obtain that relation (35): $\varphi(e^{(i)}) = x^{(i)}$, $i = 1, \ldots, n$, is sufficient to determine the mapping (34). As in the case of vector spaces, every mapping of the set $(e^{(1)}, \ldots, e^{(n)})$ into $\mathfrak{S}$ of the form $\varphi(e^{(i)}) = x^{(i)}$, $i = 1, \ldots, n$, can be extended to a unique linear mapping of $\mathbb{S}^n$ into $\mathfrak{S}$.

5.4. Basis in a symmetric quasivector space

Let $\mathfrak{S}$ be a symmetric quasivector space over $\mathbb{R}$. The set $\{c^{(i)}\}_{i=1}^k$, $c^{(i)} \in \mathfrak{S}$, $k \geqslant 1$, is a *basis* of $\mathfrak{S}$, if $c^{(i)}$ are linearly independent and $\mathfrak{S} = \mathrm{span}\{c^{(i)}\}_{i=1}^k$.

To demonstrate the selfsufficiency of the theory of quasivector spaces, we give a proof of the next proposition using entirely "quasivector" concepts.

Proposition 5. *$\mathfrak{S}$ is a symmetric quasivector space over $\mathbb{R}$. A set $\{c^{(i)}\}_{i=1}^k$, with $c^{(i)} \in \mathfrak{S}$, $k \geqslant 1$, is a basis of $\mathfrak{S}$, iff every $f \in \mathfrak{S}$ can be presented in the form (28) in a unique way (i.e. with unique scalars α_i).*

Proof. Assume first that $\{c^{(i)}\}_{i=1}^k$ is a basis of $\mathfrak{S}$, that is $\mathfrak{S} = \mathrm{span}\{c^{(i)}\}_{i=1}^k$, and every element of $\mathfrak{S}$ can be written in form (28) with suitable scalars α_i. We have to show the uniqueness of this representation, i.e.:

$$\sum_{i=1}^k \alpha_i * c^{(i)}_{\sigma(\alpha_i)} = \sum_{i=1}^k \alpha'_i * c^{(i)}_{\sigma(\alpha'_i)}, \tag{36}$$

implies $\alpha_i = \alpha'_i$ for $i = 1, \ldots, k$. Indeed, from (36), by means of the quasidistributive law, we obtain

$$\sum_{i=1}^k (\alpha_i * c^{(i)}_{\sigma(\alpha_i)} + (-\alpha'_i) * c^{(i)}_{\sigma(-\alpha'_i)}) = \sum_{i=1}^k \lambda_i * c^{(i)}_{\sigma(\lambda_i)} = 0, \tag{37}$$

where $\lambda_i = \alpha_i - \alpha'_i$. From (37), using that $\{c^{(i)}\}_{i=1}^k$ are linearly independent, we obtain $\lambda_i = \alpha_i - \alpha'_i = 0$, which proves uniqueness.

Conversely, let us suppose that the system $\{c^{(i)}\}_{i=1}^k$, $c^{(i)} \in \mathfrak{S}$, $k \geqslant 1$, is chosen in such a way that every element d of $\mathfrak{S}$ can be written in the form $d = \sum_{i=1}^k \alpha_i * c^{(i)}_{\sigma(\alpha_i)}$ with unique scalars $\alpha_i \in \mathbb{R}$. Then $\mathfrak{S} = \mathrm{span}\{c^{(i)}\}_{i=1}^k$. It remains to prove that $\{c^{(i)}\}_{i=1}^k$ are linearly independent. Indeed, assume that for some scalars α_i, $i = 1, \ldots, k$, the equality

$$\sum_{i=1}^k \alpha_i * c^{(i)}_{\sigma(\alpha_i)} = 0 \tag{38}$$

holds. Equality (38) holds for the trivial linear combination, hence $\sum_{i=1}^{k} \alpha_i * c^{(i)}_{\sigma(\alpha_i)} = \sum_{i=1}^{k} 0 * c^{(i)}$. Using the assumed uniqueness of the representation we obtain $\alpha_i = 0$ for all $i = 1, \ldots, k$, that is, (38) can only hold for a trivial linear combination. Therefore the elements $\{c^{(i)}\}_{i=1}^{k}$ are linearly independent, and hence form a basis for $\mathfrak{S}$. □

Let $\mathfrak{S}$ be a symmetric quasivector space over $\mathbb{R}$ and $\{c^{(i)}\}_{i=1}^{k}$ be a basis of $\mathfrak{S}$. Assume that $a = \sum_{i=1}^{k} \alpha_i * c^{(i)}_{\sigma(\alpha_i)}, b = \sum_{i=1}^{k} \beta_i * c^{(i)}_{\sigma(\beta_i)}$ are two elements of $\mathfrak{S}$. Their sum is

$$a + b = \sum_{i=1}^{k} \alpha_i * c^{(i)}_{\sigma(\alpha_i)} + \sum_{i=1}^{k} \beta_i * c^{(i)}_{\sigma(\beta_i)} = \sum_{i=1}^{k} (\alpha_i + \beta_i) * c^{(i)}_{\sigma(\alpha_i+\beta_i)}. \tag{39}$$

Multiplication by scalars is given by

$$\gamma * a = \sum_{i=1}^{k} |\gamma|\alpha_i * c^{(i)}_{\sigma(\alpha_i)} = \sum_{i=1}^{k} |\gamma|\alpha_i * c^{(i)}_{\sigma(|\gamma|\alpha_i)}. \tag{40}$$

To every $a = \sum_{i=1}^{k} \alpha_i * c^{(i)}_{\sigma(\alpha_i)} \in \mathfrak{S}$ we associate the vector $(\alpha_1, \alpha_2, \ldots, \alpha_k)$. Then, minding formulae (39), (40), we define addition and multiplication by scalars by means of (8), (9), arriving thus to the canonic symmetric quasivector space $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$ considered in Example 2.

As we know, negation in $\mathfrak{S}$ is the same as identity. Conjugation in $\mathfrak{S}$ coincides with opposite: $a_- = \mathrm{opp}(a) = \sum_{i=1}^{k} \alpha_i * c^{(i)}_{-\sigma(\alpha_i)} = \sum_{i=1}^{k} (-\alpha_i) * c^{(i)}_{\sigma(-\alpha_i)}$. This implies in terms of $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$, cf. (10):

$$(\alpha_1, \alpha_2, \ldots, \alpha_k)_- = \mathrm{opp}(\alpha_1, \alpha_2, \ldots, \alpha_k) = (-\alpha_1, -\alpha_2, \ldots, -\alpha_k).$$

Theorem 7. *Any symmetric quasivector space over the l.o. field of reals* $\mathbb{R}$, *with a basis of* k *elements, is isomorphic to* $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$.

Proof. Let $\mathfrak{S}$ be a symmetric quasivector space spanned over a finite basis $\{s^{(i)}\}_{i=1}^{k}$. The linear mapping $\varphi : \mathbb{S}^k \to \mathfrak{S}$, $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$, defined by

$$\varphi(\alpha_1, \alpha_2, \ldots, \alpha_k) = \alpha_1 * s^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * s^{(2)}_{\sigma(\alpha_2)} + \cdots + \alpha_k * s^{(k)}_{\sigma(\alpha_k)},$$

is a bijection. Hence φ is an isomorphism. □

Let $\mathfrak{S}$ be a symmetric quasivector space spanned over a finite basis $s^{(1)}, \ldots, s^{(k)}$. Clearly, as in the linear case, the number k of terms in the expression for the span does not change with the particular basis, hence will be called *dimension* of $\mathfrak{S}$.

As every quasivector space $\mathbb{Q}$ is a direct sum $\mathbb{Q} = \mathbb{V} \oplus \mathfrak{S}$ of a vector space $\mathbb{V}$ and a symmetric quasivector space $\mathfrak{S}$, we can speak of basis and dimension of $\mathbb{Q}$, whenever $\mathbb{V}$ and $\mathfrak{S}$ have finite bases. Namely, let $\mathbb{V} = \mathbb{V}^l$ be a l-dimensional vector space with a basis $(v^{(1)}, \ldots, v^{(l)})$ and let $\mathfrak{S} = \mathbb{S}^k$ be a k-dimensional symmetric quasivector space having a basis $(s^{(1)}, \ldots, s^{(k)})$. Then we say that $(v^{(1)}, \ldots, v^{(l)}; s^{(1)}, \ldots, s^{(k)})$ is a basis of the (l, k)-*dimensional* quasivector space $\mathbb{Q} = \mathbb{V}^l \oplus \mathbb{S}^k$.

6. Concluding remarks

We have shown that every quasivector space is a direct sum of a linear subspace and a symmetric quasivector subspace (Theorem 5). In the case of a finite basis the latter is isomorphic to the canonic symmetric quasivector space $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$. These results allow us to decompose any algebraic problem formulated in a quasivector space into two problems: a linear problem and a problem in $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$. We have also shown that the latter space is equivalent in the sense of Definition 3 to the vector space $(\mathbb{R}^k, +, \mathbb{R}, \cdot)$. Thus, practically, computation in a quasivector space is reduced to computation in vector spaces.

More specifically, assume that we have a problem formulated in a quasilinear space of monoid structure $(\mathfrak{M}, +, \mathbb{R}, *)$—one may think of $\mathfrak{M} = \mathfrak{K}$ or $\mathfrak{M} = I(\mathbb{R})$—the set of intervals on $\mathbb{R}$. To reformulate the problem in the induced quasivector space $(\mathbb{D}(\mathfrak{M}), +, \mathbb{R}, *)$ we first represent all elements $A \in \mathfrak{M}$ involved as proper elements of $\mathbb{D}(\mathfrak{M})$ of the form $a = (A, 0)$. Then, using Theorem 5, resp., formulae (24), (25), we decompose the problem into one linear problem and one symmetric quasivector problem (equivalent to a linear one), which are to be solved by means of usual techniques. What remains is the interpretation of the results in the original space of monoid structure $\mathfrak{M}$ which depends on the particular problem. The possibility to present a space of intervals as a direct sum of a vector and a symmetric quasivector space suggests that the midpoint-radius representation of intervals is appropriate for the solution of interval arithmetic (algebraic) problems. Some investigations in this direction have been performed in [9], where we confine ourselves in the semigroup of proper intervals (intervals with nonnegative radii). The present study can be applied to the algebraically more natural problem formulation involving generalized (proper and improper) intervals, cf. [4].

It is worth to mention that the meaning of Theorem 5 is transparent in the case of intervals; however this transparency is lost in the case of convex bodies (even in the plane, say a triangle). There exist substantially nonsymmetric convex bodies that are not representable as a sum of a vector and a symmetric convex body. In fact, this makes the result interesting, with an expressed methodological character. Theorem 5 permits to every generalized convex body to assign a "center" of this body. The advantage of this method is that we have a platform for easy algebraic computations. The "disadvantage" is that we should care about interpretation and visualization of the new generalized elements (similarly to what we do when using negative numbers).

In the present work we study in some detail the relation between a quasivector space $(\mathbb{Q}, +, \mathbb{R}, *)$ and the induced extended vector space $(\mathbb{Q}, +, \mathbb{R}, \cdot, _{-})$. We have demonstrated how this relation can be used in the study of symmetric quasivector spaces. This relation seems to be methodologically useful, e.g. whenever applying the theory of support functions for the solution of algebraic problems involving convex bodies. The equivalence between the quasivector space of convex bodies and the space of extended (differences of) support functions, cf. [18], shows that we can calculate with extended convex bodies in the same way as we do with extended support functions.

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