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Stochastic arithmetic: s-spaces and some applications

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It has been recently shown that computation with stochastic numbers as regard to addition and multiplication by scalars can be reduced to computation in familiar vector spaces. In this work we show how this can be used for the algebraic solution of linear systems of equations with stochastic right-hand sides. On several examples we compare the algebraic solution with the simulated solution using the CADNA package.

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1. Introduction

This work continues our study of the algebraic properties of stochastic numbers with respect to addition and multiplication by scalars [1,2,7]. A distributivity-like relation for stochastic numbers allows us to introduce spaces analogous to quasilinear spaces [5,6,8].

Stochastic numbers are Gaussian random variables with a known mean value and a known standard deviation. In practice, stochastic numbers are computed using the CESTAC method, which is a Monte-Carlo method consisting in performing each arithmetic operation several times using an arithmetic with a random rounding mode, see [3,9,11]. Some fundamental properties of stochastic numbers are considered in [4,10].

The mean values of the stochastic numbers satisfy the usual real arithmetic, whereas standard deviations are added and multiplied by scalars in a specific way: $s_1 \oplus s_2 = \sqrt{s_1^2 + s_2^2}$, $\gamma * s = |\gamma| \cdot s$. As regard to addition the system of standard deviations is an Abelian monoid with cancellation law. This monoid can be embedded in an additive group and after a suitable extension of multiplication by scalar one obtains a so-called s-space, which is closely related to a vector space [7]. This allows us to introduce in s-spaces concepts like linear combination, basis, dimension, etc. Thus computations in s-spaces are reduced to computations in vector spaces.

In sections 2 and 3 we briefly present the theory of s-spaces as regard to the arithmetic operations for addition and multiplication by scalars (for a detailed presentation of the theory, see [7]). Section 4 considers the algebraic solution of linear systems of equations which right-hand sides consist of stohastic numbers. In section 5 we present several numerical examples in order to compare the algebraic solution of a probmem involving stochastic numbers with the solution obtained by the CESTAC method.

2. Stochastic numbers and stochastic arithmetic

By $\mathbb{R}$ we denote the set of reals; we use the same notation for the linearly ordered field of reals $\mathbb{R} = (\mathbb{R}, +, \cdot, \leqslant)$. Throughout the paper $\mathbb{R}$ can be replaced by any other linearly ordered field. For any integer $n \geqslant 1$ we denote by $\mathbb{R}^n$ the set of all n-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\alpha_i \in \mathbb{R}$. The set $\mathbb{R}^n$ forms a vector space under the familiar operations of addition and multiplication by scalars denoted by $V^n = (\mathbb{R}^n, +, \mathbb{R}, \cdot)$, $n \geqslant 1$. By $\mathbb{R}^+$ we denote the set of nonnegative real numbers.

A *stochastic number* X is a Gaussian random variable with a known mean value $m \in \mathbb{R}$ and a known nonnegative standard deviation $s \in \mathbb{R}^+$ and is denoted $X = (m; s)$. The set of stochastic numbers is $\mathbb{S} = \{(m; s) \mid m \in \mathbb{R}, s \in \mathbb{R}^+\}$.

Arithmetic operations between stochastic numbers: addition and multiplication by scalars. Let $X_1 = (m_1; s_1)$, $X_2 = (m_2; s_2) \in \mathbb{S}$. Addition and multiplication by scalars are defined by

$$X_1 \,{}_s\!+ X_2 = (m_1; s_1) \,{}_s\!+ (m_2; s_2) \stackrel{\text{def}}{=} \left(m_1 + m_2;\ \sqrt{s_1^2 + s_2^2}\right), \tag{1}$$

$$\gamma \,{}_s\!* X = \gamma \,{}_s\!* (m; s) \stackrel{\text{def}}{=} \big(\gamma m; |\gamma| s\big), \quad \gamma \in \mathbb{R}. \tag{2}$$

Symmetric stochastic numbers. A stochastic number of the form $(0; s)$ is called *symmetric*. If X_1, X_2 are symmetric stochastic numbers, then $X_1 \,{}_s\!+ X_2$ and $\lambda \,{}_s\!* X_1$, $\lambda \in \mathbb{R}$, are also symmetric stochastic numbers. Clearly, there is a 1–1 correspondence between the set of symmetric stochastic numbers and the set $\mathbb{R}^+$ of standard deviations; the latter are added and multiplied by scalars according to the rules: $s_1 \oplus s_2 = \sqrt{s_1^2 + s_2^2}$, $\gamma * s = |\gamma| \cdot s$, $\gamma \in \mathbb{R}$. We use special notation "$\oplus$", "$*$" for the arithmetic operations with the standard deviations, as these operations are different from the operations for numbers. The operations "$\oplus$", "$*$" induce a special arithmetic on the set $\mathbb{R}^+$. Consider the system $(\mathbb{R}^+, \oplus, \mathbb{R}, *)$, where:

$$\alpha \oplus \beta = \sqrt{\alpha^2 + \beta^2}, \quad \alpha, \beta \in \mathbb{R}^+, \tag{3}$$

$$\gamma * \delta = |\gamma| \delta, \qquad \gamma \in \mathbb{R},\ \delta \in \mathbb{R}^+. \tag{4}$$

Proposition 1 [7]. The system $(\mathbb{R}^+, \oplus, \mathbb{R}, *)$ is an Abelian monoid with cancellation, such that for $s, t \in \mathbb{R}^+$, $\alpha, \beta \in \mathbb{R}$:

$$\alpha * (s \oplus t) = \alpha * s \oplus \alpha * t, \tag{5}$$

$$\alpha * (\beta * s) = (\alpha\beta) * s, \tag{6}$$

$$1 * s = s, \tag{7}$$

$$(-1) * s = s, \tag{8}$$

$$\sqrt{\alpha^2 + \beta^2} * s = \alpha * s \oplus \beta * s, \quad \alpha, \beta \geqslant 0. \tag{9}$$

A system satisfying the conditions of proposition 1 is called an s-space of monoid structure.

2.1. *The group system*

We extend the monoid $(\mathbb{R}^+, \oplus)$ into a group $(\mathbb{R}, \oplus)$. For $\alpha \in \mathbb{R}$ denote

$$\sigma(\alpha) = \begin{cases} +, & \text{if } \alpha \geqslant 0; \\ -, & \text{if } \alpha < 0. \end{cases}$$

We extend $\oplus$ for all $\alpha, \beta \in \mathbb{R}$:

$$\begin{aligned} \alpha \oplus \beta &= \sigma(\sigma(\alpha)\alpha^2 + \sigma(\beta)\sigma^2)\sqrt{\left|\sigma(\alpha)\alpha^2 + \sigma(\beta)\beta^2\right|} \\ &= \sigma(\alpha + \beta)\sqrt{\left|\sigma(\alpha)\alpha^2 + \sigma(\beta)\beta^2\right|}, \end{aligned} \tag{10}$$

noticing that for $\alpha, \beta \in \mathbb{R}$:

$$\sigma(\alpha + \beta) = \sigma\big(\sigma(\alpha)\alpha^2 + \sigma(\beta)\beta^2\big) = \sigma(\alpha \oplus \beta). \tag{11}$$

The system $(\mathbb{R}, \oplus)$ with $\oplus$ defined by (10) is an Abelian group with null 0 and opposite $\mathrm{opp}(\alpha) = -\alpha$, i.e. $\alpha \oplus (-\alpha) = 0$.

Examples. $1 \oplus 1 = \sqrt{2}$, $1 \oplus 2 = \sqrt{5}$, $3 \oplus 4 = 5$, $5 \oplus (-4) = 3$, $4 \oplus (-5) = -3$, $(-3) \oplus (-4) = -5$, $1 \oplus 2 \oplus 3 = \sqrt{14}$.

Formula (10) is an extension of the expression $\alpha \oplus \beta$, $\alpha, \beta \geqslant 0$, to arbitrary scalars $\alpha, \beta \in \mathbb{R}$. On the other side we can interpret (10) as an operation on standard deviations, which has been extended for generalized standard deviations (including improper, negative ones). In other words we isomorphically extend the set $\mathbb{R}^+$ of (usual, proper) standard deviations to the set $\mathbb{R}$ of generalized ones, admitting also improper (negative) standard deviations $s < 0$. Note that $(-1) * s \neq \mathrm{opp}(s) = -s$. Because of this the opposite in $(\mathbb{R}, \oplus)$ will be also denoted $\mathrm{opp}(\alpha) = \alpha_-$; we may write, e.g., $4 \oplus 5_- = 3_-$.

Multiplication by scalars is naturally extended on the set $\mathbb{R}$ of generalized standard deviations by: $\gamma * s = |\gamma| s$, $s \in \mathbb{R}$. Multiplication by -1 (negation) is denoted $\neg s = (-1) * s$. Thus in $\mathbb{R}$ we have $(-1) * s = |-1| s = s$, $s \in \mathbb{R}$.

It is easy to check that relations (5)–(9) hold true for generalized standard deviations $s, t \in \mathbb{R}$. This justifies the following definition:

Definition 1. A system $(\mathcal{S}, \oplus, \mathbb{R}, *)$, such that: (i) $(\mathcal{S}, \oplus)$ is an Abelian additive group, and (ii) for any $s, t \in \mathcal{S}$ and $\alpha, \beta \in \mathbb{R}$:

$$\alpha * (s \oplus t) = \alpha * s \oplus \alpha * t, \tag{12}$$

$$\alpha * (\beta * s) = (\alpha\beta) * s, \tag{13}$$

$$1 * s = s, \tag{14}$$

$$(-1) * s = s, \tag{15}$$

$$\sqrt{\alpha^2 + \beta^2} * s = \alpha * s \oplus \beta * s, \quad \alpha, \beta \geqslant 0, \tag{16}$$

is called an s-space over $\mathbb{R}$ (with group structure).

The canonical s-space. For any integer $k \geqslant 1$ the set $\mathcal{S} = \mathbb{R}^k$ of all k-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_k)$, where $\alpha_i \in \mathbb{R}$ and $(\alpha_1, \alpha_2, \ldots, \alpha_k) = (\beta_1, \beta_2, \ldots, \beta_k)$ whenever $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_k = \beta_k$, forms an s-space over $\mathbb{R}$ under the following operations

$$(\alpha_1, \ldots, \alpha_k) \oplus (\beta_1, \ldots, \beta_k) = (\alpha_1 \oplus \beta_1, \ldots, \alpha_k \oplus \beta_k), \tag{17}$$

$$\gamma * (\alpha_1, \alpha_2, \ldots, \alpha_k) = (|\gamma|\alpha_1, |\gamma|\alpha_2, \ldots, |\gamma|\alpha_k), \quad \gamma \in \mathbb{R}, \tag{18}$$

where $\alpha \oplus \beta$ for $\alpha, \beta \in \mathbb{R}$ is given by (10).

The s-space $\mathbf{S}^k = (\mathbb{R}^k, \oplus, \mathbb{R}, *)$ is called the *canonical s-space (of standard deviations).* Note that multiplication by -1 (negation) in $\mathbf{S}^k$ is same as identity: $\neg(\alpha_1, \ldots, \alpha_k) = (\alpha_1, \ldots, \alpha_k)$, while the opposite operator is:

$$\mathrm{opp}(\alpha_1, \alpha_2, \ldots, \alpha_k) = (\alpha_1, \alpha_2, \ldots, \alpha_k)_- = (-\alpha_1, -\alpha_2, \ldots, -\alpha_k). \tag{19}$$

We have $\mathbf{S}^k = \bigoplus_k \mathbf{S}^1$; here $\bigoplus_k$ means direct sum taken k times.

Relation (16) contains the condition $\alpha, \beta \geqslant 0$. We next give a general relation, which comprises (16) as a special case.

Denote $s_+ = s$. Since $s_- = \mathrm{opp}(s)$, the notation s_λ makes sense for any $\lambda \in \{+, -\}$.

Proposition 2 [7]. Assume that $(\mathcal{S}, \oplus, \mathbb{R}, *)$ is an s-space over $\mathbb{R}$. For all $\alpha, \beta \in \mathbb{R}$ and all $s \in \mathcal{S}$ we have

$$\sqrt{\left|\sigma(\alpha)\alpha^2 + \sigma(\beta)\beta^2\right|} * s_{\sigma(\alpha+\beta)} = \alpha * s_{\sigma(\alpha)} \oplus \beta * s_{\sigma(\beta)}. \tag{20}$$

Relation (20) can be written briefly as

$$(\alpha \oplus \beta) * s_{\sigma(\alpha+\beta)} = \alpha * s_{\sigma(\alpha)} \oplus \beta * s_{\sigma(\beta)}. \tag{21}$$

2.2. *Relations between s-spaces and vector spaces*

Vector spaces induced by s-spaces. Let $(\mathcal{S}, +, \mathbb{R}, *)$ be an s-space over $\mathbb{R}$. Define the operation "$\cdot$": $\mathbb{R} \times \mathcal{S} \longrightarrow \mathcal{S}$ by

$$\alpha \cdot c = \sqrt{|\alpha|} * c_{\sigma(\alpha)}, \quad \alpha \in \mathbb{R},\ c \in \mathcal{S}. \tag{22}$$

Equality (22) is equivalent to

$$\sigma(\alpha)\alpha^2 \cdot c = \alpha * c_{\sigma(\alpha)}, \quad \alpha \in \mathbb{R},\ c \in \mathcal{S}, \tag{23}$$

the latter meaning: $\alpha^2 \cdot c = \alpha * c$, if $\alpha \geqslant 0$ and $-\alpha^2 \cdot c = \alpha * c_-$, if $\alpha < 0$.

Proposition 3. Let $(\mathcal{S}, +, \mathbb{R}, *)$ be an s-space over $\mathbb{R}$. Then $(\mathcal{S}, +, \mathbb{R}, \cdot)$ is a vector space over $\mathbb{R}$.

The two spaces $(\mathcal{S}, +, \mathbb{R}, *)$ and $(\mathcal{S}, +, \mathbb{R}, \cdot)$ are *equivalent* in the sense that every expression in the first space can be presented in terms of the operations of the second space, and vice versa. Thus we have:

Proposition 4. Every s-space over $\mathbb{R}$ induces via (22) an equivalent vector space over $\mathbb{R}$.

S-spaces induced by vector spaces. Let $(\mathcal{S}, +, \mathbb{R}, \cdot)$ be a vector space over $\mathbb{R}$. The system $(\mathcal{S}, +, \mathbb{R}, *)$, where "$*$" is defined by

$$\alpha * c = \sigma(\alpha)\alpha^2 \cdot c_{\sigma(\alpha)} = \alpha^2 \cdot c \tag{24}$$

is an s-space over $\mathbb{R}$. The two spaces $(\mathcal{S}, +, \mathbb{R}, *)$ and $(\mathcal{S}, +, \mathbb{R}, \cdot)$ are *equivalent*.

One can make use of both operations for multiplication by scalars simultaneously. The system $(\mathcal{S}, +, \mathbb{R}, \cdot, *)$ can be viewed either as a vector space over $\mathbb{R}$ endowed with the operation (24) or as an s-space over $\mathbb{R}$ endowed via (22) with the operation "$\cdot$". In $(\mathcal{S}, +, \mathbb{R}, \cdot, *)$ one has two different notations for the opposite operator. Namely, opposite is denoted in $(\mathcal{S}, +, \mathbb{R}, \cdot)$ by $\mathrm{opp}(a) = -a$, whereas in $(\mathcal{S}, +, \mathbb{R}, *)$ one writes $\mathrm{opp}(a) = a_-$.

Using that the spaces $(\mathcal{S}, +, \mathbb{R}, \cdot)$ and $(\mathcal{S}, +, \mathbb{R}, *)$ are equivalent we can transfer familiar concepts from the theory of vector spaces to s-spaces, as shown next.

3. S-spaces of stochastic numbers

Assume that $\mathcal{S} = (\mathcal{S}, +, \mathbb{R}, *)$ is an s-space over $\mathbb{R}$ and $(\mathcal{S}, +, \mathbb{R}, \cdot)$ is the associated equivalent vector space. From the vector space $(\mathcal{S}, +, \mathbb{R}, \cdot)$ we can transfer vector space concepts, such as linear combination, linear dependence, basis, etc., to the s-space $(\mathcal{S}, +, \mathbb{R}, *)$. For example, the concept of linear combination obtains the following form.

Let $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$, $k \geqslant 1$, be finitely many (not necessarily distinct) elements of $\mathcal{S}$ and let $f = \sum_{i=1}^{k} \gamma_i \cdot c^{(i)} = \gamma_1 \cdot c^{(1)} + \gamma_2 \cdot c^{(2)} + \cdots + \gamma_k \cdot c^{(k)}$ with $\gamma_1, \gamma_2, \ldots, \gamma_k \in \mathbb{R}$ be a linear combination of $c^{(1)}, c^{(2)}, \ldots, c^{(k)}$ in the vector space $(\mathcal{S}, +, \mathbb{R}, \cdot)$. Using (22) we introduce a *linear combination* in the s-space $(\mathcal{S}, +, \mathbb{R}, *)$ by

$$f = \sqrt{|\gamma_1|} * c^{(1)}_{\sigma(\gamma_1)} + \sqrt{|\gamma_2|} * c^{(2)}_{\sigma(\gamma_2)} + \cdots + \sqrt{|\gamma_k|} * c^{(k)}_{\sigma(\gamma_k)}. \tag{25}$$

Defining α_i with $|\alpha_i| = \sqrt{|\gamma_i|}$, $\sigma(\alpha_i) = \sigma(\gamma_i)$, we can rewrite (25) as

$$f = \alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(2)}_{\sigma(\alpha_2)} + \cdots + \alpha_k * c^{(k)}_{\sigma(\alpha_k)}. \tag{26}$$

Proposition 5. The set

$$span\{c^{(1)}, c^{(2)}, \dots, c^{(k)}\} = \left\{\sum_{i=1}^{k} \alpha_i * c^{(i)}_{\sigma(\alpha_i)} \mid \alpha_i \in \mathbb{R}\right\}$$

of all linear combinations of $c^{(1)}, c^{(2)}, \dots, c^{(k)}$ is a subspace of $\mathcal{S}$.

Definition. The elements $c^{(1)}, c^{(2)}, \dots, c^{(k)} \in \mathcal{S}$, $k \geqslant 1$, are *linearly dependent* (*over* $\mathbb{R}$), if there exists a nontrivial linear combination of $\{c^{(i)}\}$, which is equal to 0, i.e. if there exist a system $\{\alpha_i\}_{i=1}^k$ with not all α_i equal to zero, such that

$$\alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(1)}_{\sigma(\alpha_2)} + \cdots + \alpha_k * c^{(k)}_{\sigma(\alpha_k)} = 0. \tag{27}$$

The elements $c^{(1)}, c^{(2)}, \dots, c^{(k)} \in \mathcal{S}$ are *linearly independent*, if (27) is possible only for the trivial linear combination, such that $\alpha_i = 0$ for all $i = 1, \dots, k$.

Linear mappings in s-spaces. Let $\mathcal{S}_1 = (\mathcal{S}_1, +, \mathbb{R}, *)$, $\mathcal{S}_2 = (\mathcal{S}_2, +, \mathbb{R}, *)$ be two s-spaces over $\mathbb{R}$ and let $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ be a linear (homomorphic) mapping, that is:

$$\varphi(x + y) = \varphi(x) + \varphi(y), \tag{28}$$

$$\varphi(\gamma * x) = \gamma * \varphi(x), \quad x, y \in \mathcal{S}_1,\ \gamma \in \mathbb{R}. \tag{29}$$

It is easy to check that $\varphi(x_-) = (\varphi(x))_-$; more generally, any linear mapping satisfies:

$$\begin{aligned}&\varphi\big(\alpha_1 * x^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * x^{(2)}_{\sigma(\alpha_2)} + \cdots + \alpha_k * x^{(k)}_{\sigma(\alpha_k)}\big)\\ &\quad = \alpha_1 * \varphi\big(x^{(1)}\big)_{\sigma(\alpha_1)} + \alpha_2 * \varphi\big(x^{(2)}\big)_{\sigma(\alpha_2)} + \cdots + \alpha_k * \varphi\big(x^{(k)}\big)_{\sigma(\alpha_k)},\end{aligned} \tag{30}$$

where $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$, $x^{(1)}, x^{(2)}, \dots, x^{(k)} \in \mathcal{S}_1$. In particular,

$$\varphi(\alpha * x_\lambda + \beta * y_\mu) = \alpha * \varphi(x)_\lambda + \beta * \varphi(y)_\mu, \quad x, y \in \mathcal{S}_1,\ \alpha, \beta \in \mathbb{R}. \tag{31}$$

Obviously condition (31) completely characterizes a linear mapping and can substitute conditions (28) and (29).

Let $(\mathcal{S}, +, \mathbb{R}, *)$ be an s-space, $x^{(1)}, x^{(2)}, \dots, x^{(n)} \in \mathcal{S}$ and let $\mathbf{S}^n = (\mathbb{R}^n, \oplus, \mathbb{R}, *)$ be the canonical s-space. The mapping $\varphi : \mathbf{S}^n \to \mathcal{S}$, such that

$$\varphi(\alpha_1, \alpha_2, \dots, \alpha_n) = \alpha_1 * x^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * x^{(2)}_{\sigma(\alpha_2)} + \cdots + \alpha_n * x^{(n)}_{\sigma(\alpha_n)}, \tag{32}$$

is linear [7].

Denote the n-vector $e^{(i)} = (0, 0, \dots, 0, 1, 0, \dots, 0)$, where the component 1 is on the ith place. We consider $e^{(i)}$ as element of $\mathbf{S}^n$, where $\mathrm{opp}(e^{(i)}) = e^{(i)}_-$ and $\neg e^{(i)} = e^{(i)}$. Relation (32) implies

$$\varphi\big(e^{(i)}\big) = \alpha_i * x^{(i)}_{\sigma(\alpha_i)}\big|_{\alpha_i = 1} = x^{(i)}, \quad i = 1, \dots, n. \tag{33}$$

The mapping φ is the only linear mapping from $\mathbf{S}^n$ to $\mathcal{S}$ with the property (33). Indeed, if (33) holds, then by (30),

$$\varphi(\alpha_1, \alpha_2, \ldots, \alpha_n) = \varphi\Big(\sum \alpha_i * e^{(i)}_{\sigma(\alpha_i)}\Big) \\ = \sum \alpha_i * \varphi\big(e^{(i)}\big)_{\sigma(\alpha_i)} = \sum \alpha_i * x^{(i)}_{\sigma(\alpha_i)}.$$

We thus obtain that relation (33): $\varphi(e^{(i)}) = x^{(i)}, i = 1, \ldots, n$, is sufficient to determine the mapping (32). As in the case of vector spaces, every mapping of the set $(e^{(1)}, \ldots, e^{(n)})$ into $\mathcal{S}$ of the form $\varphi(e^{(i)}) = x^{(i)}, i = 1, \ldots, n$, can be extended to a unique linear mapping of $\mathbf{S}^n$ into $\mathcal{S}$.

Basis in an s-space. Let $\mathcal{S}$ be an s-space over $\mathbb{R}$. The set $\{c^{(i)}\}_{i=1}^k$, $c^{(i)} \in \mathcal{S}$, $k \geqslant 1$, is a *basis* of $\mathcal{S}$, if $c^{(i)}$ are linearly independent and $\mathcal{S} = \operatorname{span}\{c^{(i)}\}_{i=1}^k$.

Proposition 6. A set $\{c^{(i)}\}_{i=1}^k$, $c^{(i)} \in \mathcal{S}$, $k \geqslant 1$, is a basis of $\mathcal{S}$, iff every $f \in \mathcal{S}$ can be presented in the form (26) in a unique way (i.e. with unique scalars α_i).

Let $\mathcal{S}$ be an s-space over $\mathbb{R}$ and $\{c^{(i)}\}_{i=1}^k$ be a basis of $\mathcal{S}$. Assume that $a = \sum_{i=1}^k \alpha_i * c^{(i)}_{\sigma(\alpha_i)}$, $b = \sum_{i=1}^k \beta_i * c^{(i)}_{\sigma(\beta_i)}$ are two elements of $\mathcal{S}$. Their sum is

$$a + b = \sum_{i=1}^k \alpha_i * c^{(i)}_{\sigma(\alpha_i)} + \sum_{i=1}^k \beta_i * c^{(i)}_{\sigma(\beta_i)} = \sum_{i=1}^k (\alpha_i \oplus \beta_i) * c^{(i)}_{\sigma(\alpha_i + \beta_i)}. \tag{34}$$

Multiplication by scalars is given by

$$\gamma * a = \sum_{i=1}^k |\gamma|\alpha_i * c^{(i)}_{\sigma(\alpha_i)} = \sum_{i=1}^k |\gamma|\alpha_i * c^{(i)}_{\sigma(|\gamma|\alpha_i)}. \tag{35}$$

To every $a = \sum_{i=1}^k \alpha_i * c^{(i)}_{\sigma(\alpha_i)} \in \mathcal{S}$ we associate the k-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k)$. Then, minding formulae (34), (35), we define addition and multiplication by scalars by means of (17), (18), arriving thus to the canonic s-space $\mathbf{S}^k = (\mathbb{R}^k, \oplus, \mathbb{R}, *)$.

As we know, negation in $\mathcal{S}$ is same as identity. Opposite in $\mathcal{S}$ is $a_- = \operatorname{opp}(a) = \sum_{i=1}^k \alpha_i * c^{(i)}_{-\sigma(\alpha_i)} = \sum_{i=1}^k (-\alpha_i) * c^{(i)}_{\sigma(-\alpha_i)}$, or, in terms of $\mathbf{S}^k = (\mathbb{R}^k, \oplus, \mathbb{R}, *)$ we obtain (19).

Theorem 1 [7]. Any s-space over $\mathbb{R}$, with a basis of k elements, is isomorphic to $\mathbf{S}^k$.

Let $\mathcal{S}$ be an s-space spanned over a finite basis $s^{(1)}, s^{(2)}, \ldots, s^{(k)}$. As in the vector case, the number k does not change with the particular basis and will be called *dimension* of $\mathcal{S}$.

Stochastic numbers $(m; s)$ can be considered as elements of a direct sum $\mathcal{V} \oplus \mathcal{S}$ of a vector space $\mathcal{V}$ and an s-space $\mathcal{S}$. Assume that $\mathcal{V}$ and $\mathcal{S}$ have finite bases both of dimension k. Namely, let $\mathcal{V} = \mathbf{V}^k$ be a k-dimensional vector space with a basis

$(v^{(1)}, \ldots, v^{(k)})$ and let $\mathcal{S} = \mathbf{S}^k$ be a k-dimensional s-space having a basis $(s^{(1)}, \ldots, s^{(k)})$. Then we say that $(v^{(1)}, \ldots, v^{(k)}; s^{(1)}, \ldots, s^{(k)})$ is a basis of the *k-dimensional* space $\mathbf{V}^k \oplus \mathbf{S}^k$. Such a setting allows us to consider numerical problems involving vectors and matrices, wherein the numeric variables have been substituted by stochastic ones. In the next section we consider such a problem.

4. Linear systems with stochastic right-hand side

We consider a linear system $Ax = b$, such that A is a real $n \times n$-matrix and the right-hand side b is a vector of stochastic numbers. Then the solution vector x also consists of stochastic numbers, and, respectively, all arithmetic operations (additions and multiplications by scalars) in the expression Ax are interpreted in the sense of the above presented theory (therefore we shall write $A \,{}_s\!{*}\, x$ instead of Ax).

Problem. Assume that $A = (\alpha_{ij})_{i,j=1}^n$, $\alpha_{ij} \in \mathbb{R}$, is a real $n \times n$-matrix, and $b = (b'; b'')$ is an n-vector of (generalized) stochastic numbers, such that $b', b'' \in \mathbb{R}^n$, $b' = (b'_1, \ldots, b'_n)$, $b'' = (b''_1, \ldots, b''_n)$. We look for a (generalized) stochastic vector $x = (x'; x'')$, $x', x'' \in \mathbb{R}^n$, that is an n-vector of stochastic numbers, such that $A \,{}_s\!{*}\, x = b$.

Solution. Clearly, the system $A \,{}_s\!{*}\, x = b$ reduces to a linear system $Ax' = b'$ for the vector $x' = (x'_1, \ldots, x'_n)$ of mean values and a system $A * x'' = b''$ for the vector $x'' = (x''_1, \ldots, x''_n)$ of standard deviations.

The elements of the vector $A * x''$ are $c_i = \alpha_{i1} * x''_1 \oplus \cdots \oplus \alpha_{in} * x''_n = \sqrt{\alpha_{i1}^2 x_1''^2 + \cdots + \alpha_{in}^2 x_n''^2}$, $i = 1, \ldots, n$.

Setting $\operatorname{sign}(x''_i)(x''_i)^2 = y_i$, $\operatorname{sign}(b''_i)(b''_i)^2 = c_i$, we obtain a linear $n \times n$ system $Dy = c$ for $y = (y_i)$, where $D = (\alpha_{ij}^2)$. If D is nonsingular we can solve the system $Dy = c$ for the vector y, and then obtain the standard deviation vector x by means of $x_i = \operatorname{sign}(y_i)\sqrt{|y_i|}$. Thus for the solution of the original problem it is necessary and sufficient that both matrices $A = (\alpha_{ij})$ and $D = (\alpha_{ij}^2)$ are nonsingular.

5. Comparison of stochastic algebraic solutions with solutions obtained by the CESTAC method

Numerical experiments

Several experiments have been performed in order to compare the theory and the results obtained with the CESTAC method for impresise data. A scalar product AB has been computed when the vector B is a real vector and the components of the vector A are stochastic numbers. As it is shown in the theory of the CESTAC method, such a stochastic number can be represented by an n-tuple of random values with a known mean value m and a known standard deviation σ. In our examples $n = 10$, but it is enough to take $n = 3$ as implemented in the CADNA software [3,4].

Table 1
Percentages of theoretical standard deviation $\sigma\sqrt{N}$ outside the confidence interval

$N \setminus k$	3	4	5	6	7	10
10	13.2	5.1	3.6	1.7	0.9	0.2
100	11.6	5.2	3.5	2.1	1.0	0.3
1000	11.4	6.2	3.4	2.0	1.1	0.3
10000	12.4	5.4	2.5	2.3	1.3	0.1

We consider an example of N components of A with $n = 3$ and a Gaussian random number generator with $m = 1$ and $\sigma = 0.001$.

With the above conditions ($m = 1, \sigma = 0.001$) the scalar product AB has been computed k times for various sizes $N = 10, 100, \dots, 100\,000$. For each size N the mean value $\overline{\delta}$ of the standard deviation δ_i of the result ($i = 1, 2, \dots, k$) has been computed as well as the standard deviation τ of the δ_i, i.e.

$$\overline{\delta} = \frac{1}{k}\sum_{i=1}^{k}\delta_i, \qquad \tau^2 = \frac{1}{k-1}\sum_{i=1}^{k}\left(\delta_i - \overline{\delta}\right)^2.$$

This provides samples of size k whose mean values approximate the theoretical standard deviation.

As all the components of A have the same distribution, from the central limit theorem the distribution of the scalar product is approximately Gaussian. So the theoretical value of the standard deviation of the scalar product $\sigma\sqrt{N}$ should be in the interval $[\overline{\delta} - 2\tau, \overline{\delta} + 2\tau]$ with a probability 0.95.

Table 1 reports the percentages of cases where the theoretical standard deviation $\sigma\sqrt{N}$ is outside the computed confidence interval. These percentages have been computed with 1000 runs.

Comments. From table 1, it is clear a posteriori that the distribution of the scalar product is effectivelly Gaussian, as a size of 4 to 5 for the samples is enough to approximate the theoretical value, whereas if it were not the samples should rather be of size 30.

6. Conclusion

In this work we briefly outline the algebraic theory of stochastic numbers related to the operations addition and multiplication by scalar and apply this theory for the solution of a linear algebraic problem.

The theoretic study of the properties of stochastic numbers allow us to obtain rigorous abstract definition of stochastic numbers with respect to the operations addition and multiplication by scalars. Our theory also allows us to solve algebraic problems with stochastic numbers. This gives us a possibility to compare algebraically obtained results with practical applications of stochastic numbers, such as the ones provided by the CESTAC method [3]. Such comparisons will give additional information related to the stochastic behaviour of random roundings in the course of numerical computations.

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