Stochastic Arithmetic: Addition and Multiplication by Scalars

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Abstract

Stochastic arithmetic involving addition and multiplication by scalars is studied with an emphasis on the abstract structure of the set of stochastic numbers. New properties of stochastic numbers are obtained such as a special distributivity relation corresponding to the second distributivity law in a vector space. This allows us to introduce algebraic systems abstracting properties of stochastic numbers, with respect to addition and multiplication by scalars. We define axiomatically such systems with group structure and give them a complete characterization in the finite dimensional case. This permits to reduce computation with stochastic numbers to computation in familiar vector spaces.

Key words: stochastic numbers, stochastic arithmetic, standard deviations, variances, S-space.

1 Introduction

This work continues our study of the algebraic properties of stochastic numbers [1], [2]. We pay special attention to the algebraic properties of stochastic numbers with respect to the operations addition and multiplication by scalars. A new distributivity relation for stochastic numbers which corresponds to the

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second distributivity law in a vector space allows us to introduce spaces analogous to quasilinear spaces which appear in convex analysis [6], [11], and interval analysis [8], [10].

Stochastic numbers are gaussian random variables with a known mean value and a known standard deviation. The set of operators on stochastic numbers is defined as stochastic arithmetic. Stochastic numbers and stochastic arithmetic are tightly connected to Jean Vignes' CESTAC method [15], for the following reason.

Stochastic numbers can be computed in practice using the CESTAC method, which consists in performing several times each floating point operation with a random rounding mode. Thus several samples representing the same mathematical result are obtained and it has been proved that these samples generally have a gaussian distribution. So a close estimation of the mean value, considered as the exact result and of the standard deviation can be computed using the classical statistical tools, see [3], [13], [14]. In fact, the CESTAC method can be considered as a discretization of the computation with stochastic arithmetic or conversely and more correctly, stochastic arithmetic and stochastic numbers can be viewed as a continuous modelization of the CESTAC method. Some fundamental properties of stochastic numbers can be found in [4], [5], [12], [15].

As the mean values of the stochastic numbers obey the usual real arithmetic we concentrate on symmetric stochastic numbers (such with mean value zero), that is on the arithmetic for standard deviations. Standard deviations are added and multiplied by scalars in a specific way: $s_1 \oplus s_2 = \sqrt{s_1^2 + s_2^2}$, $\gamma * s = |\gamma| \cdot s$. As regard to these operations the system of standard deviations is an abelian monoid with cancellation satisfying certain relations for the multiplication by scalars. In this work we embed this system in an additive group obtaining thus a system here called an S-space. We point out a relation between S-spaces and vector spaces. Using this relation we introduce in S-spaces certain concepts characteristic for vector spaces, such as linear combination, basis, dimension etc. Thus computations in S-spaces are reduced to computations in vector spaces.

Section 2 is devoted to stochastic numbers and the two arithmetic operations for (symmetric) stochastic numbers: addition and multiplication by scalars. Section 3 considers a distributivity relation for symmetric stochastic numbers. Section 4 is devoted to generalized symmetric stochastic numbers forming an additive group. In Section 5 we discuss the relation between S-spaces and vector spaces. On the base of this relation a theory of S-spaces is outlined in Sections 6 and 7.

2 Stochastic Numbers and Stochastic Arithmetic

By $\mathbb{R}$ we denote the set of reals; we use the same notation for the linearly ordered field of reals $\mathbb{R} = (\mathbb{R}, +, \cdot, \leq)$. Throughout the paper $\mathbb{R}$ can be replaced by any other linearly ordered field. For any integer $n \geq 1$ we denote by $\mathbb{R}^n$ the set of all n-tuples $(\alpha_1, \alpha_2, ..., \alpha_n)$, where $\alpha_i \in \mathbb{R}$. The set $\mathbb{R}^n$ forms a vector space under the operations of addition and multiplication by scalars denoted by $V^n = (\mathbb{R}^n, +, \mathbb{R}, \cdot)$, $n \geq 1$. By $\mathbb{R}^+$ we denote the set of nonnegative reals.

A *stochastic number* X is a gaussian random variable with a known mean value m and a known standard deviation s and is denoted $X = (m; s)$. The set of stochastic numbers is denoted $\mathbb{S} = \{(m; s) \mid m \in \mathbb{R}, s \in \mathbb{R}^+\}$.

Arithmetic operations between stochastic numbers: addition and multiplication by scalars. Let $X_1 = (m_1; s_1)$, $X_2 = (m_2; s_2) \in \mathbb{S}$. (Usual) equality between two stochastic numbers X_1, X_2 is: $X_1 = X_2$, if $m_1 = m_2$ and $s_1 = s_2$. In this work we concentrate on the operations addition and multiplication by scalars respectively noted $_{\mathrm{s}}+$ and $_{\mathrm{s}}*$:

The definition of these operators are identical to those of addition of two independant gaussian functions and of the multiplication of a gaussian function by a scalar, i.e. :

$$X_1 \ _{\mathrm{s}}+ \ X_2 = (m_1; s_1) \ _{\mathrm{s}}+ \ (m_2; s_2) \overset{def}{=} \left(m_1 + m_2;\ \sqrt{s_1^2 + s_2^2}\right), \tag{1}$$

$$\gamma \ _{\mathrm{s}}* \ X = \gamma \ _{\mathrm{s}}* \ (m; s) \overset{def}{=} \left(\gamma m;\ |\gamma| s\right),\ \gamma \in \mathbb{R}. \tag{2}$$

The mean values m of the stochastic numbers satisfy the familiar vector space axioms. We shall concentrate on the standard deviations and their properties.

Symmetric stochastic numbers. A stochastic number of the form $(0; s)$ is called *symmetric*. If X_1, X_2 are symmetric stochastic numbers, then $X_1 \ _{\mathrm{s}}+ \ X_2$ and $\lambda \ _{\mathrm{s}}* \ X_1$, $\lambda \in \mathbb{R}$, are also symmetric stochastic numbers. Clearly, there is a one to one correspondence between the set of symmetric stochastic numbers and the set $\mathbb{R}^+$ of standard deviations. Moreover the set of symmetric stochastic is a subset of the set of stochastic zeroes as defined in ([15]).

Addition (1) and multiplication by scalars (2) for symmetric stochastic numbers $X = (0; s)$, $s \in \mathbb{R}^+$, are:

$$
\begin{aligned}
X_1 \,{}_s\!+\, X_2 &= (0; s_1) \,{}_s\!+\, (0; s_2) = (0; \sqrt{s_1^2 + s_2^2}),\\
\gamma \,{}_s\!*\, X &= \gamma \,{}_s\!*\, (0; s) = (0; |\gamma| s), \ \gamma \in \mathbb{R}.
\end{aligned}
$$

Thus standard deviations $s \in \mathbb{R}^+$ are added and multiplied by scalars according to the rules: $s_1 \oplus s_2 = \sqrt{s_1^2 + s_2^2}$, $\gamma * s = |\gamma| \cdot s,\ \gamma \in \mathbb{R}$. Here and in the sequel we use special notation "$\oplus$", "$*$" for the two arithmetic operations between the standard deviations of stochastic numbers, as these operations are different from the operations for numbers. The operations "$\oplus$", "$*$" induce a special arithmetic on the set $\mathbb{R}^+$ of nonnegative numbers, which we shall study in some detail in the sequel. Thus, consider the system $(\mathbb{R}^+, \oplus, \mathbb{R}, *)$, where:

$$
\alpha \oplus \beta = \sqrt{\alpha^2 + \beta^2}, \ \alpha, \beta \in \mathbb{R}^+, \tag{3}
$$

$$
\gamma * \delta = |\gamma| \delta, \ \gamma \in \mathbb{R}, \delta \in \mathbb{R}^+. \tag{4}
$$

Setting $\gamma = \alpha^2$, $\delta = \beta^2$, we can write (3) in the form

$$
\sqrt{\gamma} \oplus \sqrt{\delta} = \sqrt{\gamma + \delta}, \ \gamma, \delta \in \mathbb{R}^+. \tag{5}
$$

For example: $1 \oplus 1 = \sqrt{2}$, $1 \oplus 2 = \sqrt{5}$, $3 \oplus 4 = 5$, $\sqrt{3} \oplus \sqrt{5} = \sqrt{8}$, $(-1) * 4 = 4$, $(-2) * 4 = 8$, etc. Note that $s \oplus s = \sqrt{2} s = \sqrt{2} * s$.

Proposition 1 *The system $(\mathbb{R}^+, \oplus, \mathbb{R}, *)$ is an abelian monoid with cancellation, such that for $s, t \in \mathbb{R}^+$, $\alpha, \beta \in \mathbb{R}$:*

$$
\alpha * (s \oplus t) = \alpha * s \oplus \alpha * t, \tag{6}
$$

$$
\alpha * (\beta * s) = (\alpha\beta) * s, \tag{7}
$$

$$
1 * s = s, \tag{8}
$$

$$
(-1) * s = s. \tag{9}
$$

Proof. From $\alpha \oplus (\beta \oplus \gamma) = \alpha \oplus \sqrt{\beta^2 + \gamma^2} = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$ and $(\alpha \oplus \beta) \oplus \gamma = \sqrt{\alpha^2 + \beta^2} \oplus \gamma = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$ we conclude that $(\mathbb{R}^+, \oplus)$ is a semigroup. As $\alpha \oplus 0 = \sqrt{\alpha^2} = \alpha$, we see that $(\mathbb{R}^+, \oplus)$ is a monoid with neutral element 0. Commutativity is obvious. Cancellation law: $\alpha \oplus x = \beta \oplus x$ implies $\sqrt{\alpha^2 + x^2} = \sqrt{\beta^2 + x^2}$, i. e. $\alpha^2 = \beta^2$, that is $\alpha = \beta$. The equalities $\alpha * s \oplus \alpha * t = |\alpha| s \oplus |\alpha| t = \sqrt{\alpha^2 s^2 + \alpha^2 t^2} = |\alpha| \sqrt{s^2 + t^2} = \alpha * (s \oplus t)$ prove (6). The rest is obvious. $\square$

Note that multiplication by -1 (also called negation) satisfies for $s \in \mathbb{R}^+$: $(-1) * s = |-1| s = s$. Thus, negation coincides with identity, whereas in a vector space (such as the vector space of mean values) it coincides with opposite.

Note that none of the relations (6)–(9) indicates for a possibility to factor out a common multiplier s in an expression of the form $\alpha * s \oplus \beta * s$. We need a relation that corresponds to the familiar second distributive law in a vector space: $(\alpha+\beta)c = \alpha c + \beta c$. We shall derive such a relation in the next section.

3 Distributivity Relation for Stochastic Numbers

Remember that as mentioned earlier we only deal here with standard-deviations as mean-values satisfy the vector space axioms.

Proposition 2 *For $\alpha, \beta \geq 0,\ s \in \mathbb{R}^+$, we have:*

$$\sqrt{\alpha^2+\beta^2} * s = \alpha * s \oplus \beta * s, \tag{10}$$

or, equivalently, for $\gamma, \delta \geq 0,\ s \in \mathbb{R}^+$:

$$\sqrt{\gamma+\delta} * s = \sqrt{\gamma} * s \oplus \sqrt{\delta} * s. \tag{11}$$

Proof. Using (3), (4) we have:

$$\begin{aligned}\alpha * s \oplus \beta * s &= |\alpha| s \oplus |\beta| s = \sqrt{(\alpha s)^2 + (\beta s)^2} \\ &= \sqrt{\alpha^2+\beta^2}s = \sqrt{\alpha^2+\beta^2} * s.\end{aligned}$$

To obtain (11) set $\gamma = \alpha^2$, $\delta = \beta^2$ in (10). □

Remark. We can write (10) in the form $(\alpha \oplus \beta) * s = \alpha * s \oplus \beta * s$. However, note that the "$\oplus$" in the left-hand side of this equality is a shorthand notation for an expression in the field of scalars, namely $\alpha \oplus \beta = \sqrt{\alpha^2+\beta^2}$, whereas the "$\oplus$" in the right-hand side is the additive operation in the set of standard deviations.

Let us note that relation (10) implies $0 * s = 0$. Indeed, setting $\alpha = 0, \beta = 1$ in (10) we obtain $1 * s = 0 * s + 1 * s$, that is $0 * s = 0$.

Combining proposition 1 and proposition 2 we obtain:

Proposition 3 *The system $(\mathbb{R}^+, \oplus, \mathbb{R}, *)$ is an abelian monoid with cancellation, such that for $s, t \in \mathbb{R}^+$, $\alpha, \beta \in \mathbb{R}$:*

$$
\begin{aligned}
\alpha * (s \oplus t) &= \alpha * s \oplus \alpha * t, && (12)\\
\alpha * (\beta * s) &= (\alpha\beta) * s, && (13)\\
1 * s &= s, && (14)\\
(-1) * s &= s, && (15)\\
\sqrt{\alpha^2 + \beta^2} * s &= \alpha * s \oplus \beta * s,\ \alpha, \beta \geq 0. && (16)
\end{aligned}
$$

A system satisfying the conditions of proposition 3 will be further called an S-space of monoid structure. Namely, we have (below we denote the additive operation just by "+"):

Definition 4 *A system* $(\mathbb{R}^+, +, \mathbb{R}, *)$ *is called an S-space with monoid structure if* $(\mathbb{R}^+, +)$ *is an abelian monoid with cancellation, and for* $s, t \in \mathbb{R}^+$, $\alpha, \beta \in \mathbb{R}$*:*

$$
\begin{aligned}
\alpha * (s + t) &= \alpha * s + \alpha * t, && (17)\\
\alpha * (\beta * s) &= (\alpha\beta) * s, && (18)\\
1 * s &= s, && (19)\\
(-1) * s &= s, && (20)\\
\sqrt{\alpha^2 + \beta^2} * s &= \alpha * s + \beta * s,\ \alpha, \beta \geq 0. && (21)
\end{aligned}
$$

Representation by variances. Stochastic numbers can be represented alternatively by means of mean-values and variances $v = s^2$, i. e. as pairs of the form: $X = (m; v)$. Then the sum of two stochastic numbers $(m_1; v_1)$ and $(m_2; v_2)$ is $(m_1 + m_2; v_1 + v_2)$, whereas the product of $(m; v)$ by a scalar $\gamma \in \mathbb{R}$ is $(\gamma m; \gamma^2 v)$. Hence variances are added as usually but are multiplied by scalars according to the formula

$$\gamma \diamond v = \gamma^2 v, \quad \gamma \in \mathbb{R}, \tag{22}$$

e. g. $\sqrt{2} \diamond v = 2v$, $(-2) \diamond v = 4v$, etc.

Proposition 5 *The system* $(\mathbb{R}^+, +, \mathbb{R}, \diamond)$ *is an S-space of monoid structure.*

Proof. We have to check that relations (17)–(21) are satisfied. Consider for example (21). We have $\alpha \diamond v + \beta \diamond v = \alpha^2 v + \beta^2 v = (\alpha^2 + \beta^2)v = \sqrt{\alpha^2 + \beta^2} \diamond v$. $\square$

Note that $v + v = 2v = \sqrt{2} \diamond v$.

4 The Group System

For $\alpha \in \mathbb{R}$ denote $\sigma(\alpha) = \{+, \alpha \geq 0; -, \alpha < 0\}$. We shall use $\sigma(\alpha)$ instead of $\mathrm{sign}(\alpha)$ if placed in front of a number, e. g. $\sigma(\alpha)\alpha^2$ is same as $\mathrm{sign}(\alpha)\alpha^2$.

It is convenient to define in $\mathbb{R}$ a "signed square root":

$$\sqrt{\gamma}^{\circ} = \sigma(\gamma)\sqrt{|\gamma|},\ \gamma \in \mathbb{R}. \tag{23}$$

Thus, $\sqrt{\gamma}^{\circ} = \sqrt{\gamma}$ for $\gamma \geq 0$, but $\sqrt{\gamma}^{\circ} = -\sqrt{|\gamma|}$ for $\gamma < 0$. Note that the values of the signed square root $\sqrt{\gamma}^{\circ}$ for $\gamma < 0$ are negative real numbers and not complex numbers, e. g. $\sqrt{-4}^{\circ} = -2$.

We now extend (3) for all $\alpha, \beta \in \mathbb{R}$:

$$\alpha \oplus \beta = \sqrt{\sigma(\alpha)\alpha^2 + \sigma(\beta)\beta^2}^{\circ}, \tag{24}$$

or, in usual terms:

$$\begin{aligned} \alpha \oplus \beta &= \sigma(\sigma(\alpha)\alpha^2 + \sigma(\beta)\sigma^2)\sqrt{|\sigma(\alpha)\alpha^2 + \sigma(\beta)\beta^2|} \\ &= \sigma(\alpha + \beta)\sqrt{|\sigma(\alpha)\alpha^2 + \sigma(\beta)\beta^2|}, \end{aligned}$$

noticing that for $\alpha, \beta \in \mathbb{R}$:

$$\sigma(\alpha + \beta) = \sigma(\sigma(\alpha)\alpha^2 + \sigma(\beta)\beta^2) = \sigma(\alpha \oplus \beta). \tag{25}$$

Proposition 6 *The system* $(\mathbb{R}, \oplus)$*, where "*$\oplus$*" is defined by (24), is an abelian group with null* 0 *and opposite* $opp(\alpha) = -\alpha$*, i. e.* $\alpha \oplus (-\alpha) = 0$.

In other words $(\mathbb{R}^+, \oplus)$ is embedded isomorphically in $(\mathbb{R}, \oplus)$.

Example 7 $1 \oplus 1 = \sqrt{2}$, $1 \oplus 2 = \sqrt{5}$, $3 \oplus 4 = 5$, $5 \oplus (-4) = 3$, $4 \oplus (-5) = -3$, $(-3) \oplus (-4) = -5$, $1 \oplus 2 \oplus 3 = \sqrt{14}$.

Addition of generalized standard deviations. We can look at formula (24) as an extension of the expression $\alpha \oplus \beta$, $\alpha, \beta \geq 0$, to arbitrary scalars $\alpha, \beta \in \mathbb{R}$. On the other side we can interpret (24) as an operation on standard deviations, which has been now extended for generalized standard deviations (including improper, negative ones). In other words we isomorphically extend the set $\mathbb{R}^+$ of (usual, proper) standard deviations to the set $\mathbb{R}$ of generalized ones, admitting also improper (negative) standard deviations $s < 0$. The opposite in $(\mathbb{R}, \oplus)$ will be denoted $\mathrm{opp}(\alpha) = \alpha_-$.

Multiplication by scalars. Multiplication by scalars is extended on the set $\mathbb{R}$ of generalized standard deviations by: $\gamma * s = |\gamma| s$, $s \in \mathbb{R}$. Multiplication by -1 (negation) will be denoted $\neg s = (-1) * s$. Thus in $\mathbb{R}$ we have $(-1) * s = |-1| s = s$, $s \in \mathbb{R}$.

It is easy to check that relations (17)–(21) hold true for generalized standard deviations $s, t \in \mathbb{R}$. This justifies the following definition:

Definition 8 *A system* $(\mathbf{S}, \oplus, \mathbb{R}, *)$*, such that: i)* $(\mathbf{S}, \oplus)$ *is an abelian additive group, and ii) for any* $s, t \in \mathbf{S}$ *and* $\alpha, \beta \in \mathbb{R}$*:*

$$\begin{aligned} \alpha * (s \oplus t) &= \alpha * s \oplus \alpha * t, & (26)\\ \alpha * (\beta * s) &= (\alpha\beta) * s, & (27)\\ 1 * s &= s, & (28)\\ (-1) * s &= s, & (29)\\ \sqrt{\alpha^2 + \beta^2} * s &= \alpha * s \oplus \beta * s, \ \ \alpha, \beta \geq 0, & (30) \end{aligned}$$

is called an S-space over $\mathbb{R}$ *with group structure or just an S-space over* $\mathbb{R}$*.*

Example 9 *For any integer* $k \geq 1$ *the set* $\mathbf{S} = \mathbb{R}^k$ *of all* k*-tuples* $(\alpha_1, \alpha_2, ..., \alpha_k)$*, where* $\alpha_i \in \mathbb{R}$ *and* $(\alpha_1, \alpha_2, ..., \alpha_k) = (\beta_1, \beta_2, ..., \beta_k)$ *whenever* $\alpha_1 = \beta_1, \alpha_2 = \beta_2, ..., \alpha_k = \beta_k$*, forms an S-space over* $\mathbb{R}$ *under the following operations*

$$\begin{aligned} (\alpha_1, ..., \alpha_k) \oplus (\beta_1, ..., \beta_k) &= (\alpha_1 \oplus \beta_1, ..., \alpha_k \oplus \beta_k), & (31)\\ \gamma * (\alpha_1, \alpha_2, ..., \alpha_k) &= (|\gamma|\alpha_1, |\gamma|\alpha_2, ..., |\gamma|\alpha_k), \ \ \gamma \in \mathbb{R}, & (32) \end{aligned}$$

where $\alpha \oplus \beta = \sqrt{\sigma(\alpha)\alpha^2 + \sigma(\beta)\beta^2}^{\circ}$*, cf. (24).*

The S-space $\mathbf{S}^k_{(s)} = (\mathbb{R}^k, \oplus, \mathbb{R}, *)$ *will be called the* canonical S-space of standard deviations. *Note that multiplication by* -1 *(negation) in* $\mathbf{S}^k_{(s)}$ *is same as identity:* $\neg(\alpha_1, ..., \alpha_k) = (\alpha_1, ..., \alpha_k)$*, while the opposite operator is:*

$$opp(\alpha_1, \alpha_2, ..., \alpha_k) = (\alpha_1, \alpha_2, ..., \alpha_k)_- = (-\alpha_1, -\alpha_2, ..., -\alpha_k). \qquad (33)$$

We have $\mathbf{S}^k_{(s)} = \bigoplus_k \mathbf{S}^1_{(s)}$*; here* $\bigoplus_k$ *means direct sum taken* k *times.*

Example 10 *For any integer* $k \geq 1$ *the set* $S = \mathbb{R}^k$ *of all* k*-tuples* $(\alpha_1, \alpha_2, ..., \alpha_k)$*, where* $\alpha_i \in \mathbb{R}$ *and* $(\alpha_1, \alpha_2, ..., \alpha_k) = (\beta_1, \beta_2, ..., \beta_k)$ *if and only if* $\alpha_1 = \beta_1, \alpha_2 = \beta_2, ..., \alpha_k = \beta_k$*, forms an S-space over* $\mathbb{R}$ *under the following operations*

$$\begin{aligned} (\alpha_1, \alpha_2, ..., \alpha_k) + (\beta_1, \beta_2, ..., \beta_k) &= (\alpha_1 + \beta_1, ..., \alpha_k + \beta_k), & (34)\\ \gamma \diamond (\alpha_1, \alpha_2, ..., \alpha_k) &= (\gamma^2\alpha_1, \gamma^2\alpha_2, ..., \gamma^2\alpha_k), \ \ \gamma \in \mathbb{R}. & (35) \end{aligned}$$

This S-space will be denoted by $\mathbf{S}^k_{(v)} = (\mathbb{R}^k, +, \mathbb{R}, \diamond)$ *and called the* canonical S-space of variances*. Negation is the same as identity and opposite is (33). We have* $\mathbf{S}^k_{(v)} = \bigoplus_k \mathbf{S}^1_{(v)}$.

Rules for computation in an S-space. Assume that $\mathbf{S}, \oplus, \mathbb{R}, *)$ is an S-space.

Relation (30) contains the condition $\alpha, \beta \geq 0$ and thus is not convenient for algebraic computations. We shall next derive an unconditional relation, which will imply (30) as a special case.

Denote $s_+ = s$. Since $s_- = \mathrm{opp}(s)$, the notation s_λ makes sense for any $\lambda \in \{+, -\}$.

Proposition 11 *Assume that* $(\mathbf{S}, \oplus, \mathbb{R}, *)$ *is an S-space over* $\mathbb{R}$*. For all* $\alpha, \beta \in \mathbb{R}$ *and all* $s \in \mathbf{S}$ *we have*

$$\sqrt{|\sigma(\alpha)\alpha^2 + \sigma(\beta)\beta^2|} * s_{\sigma(\alpha+\beta)} = \alpha * s_{\sigma(\alpha)} \oplus \beta * s_{\sigma(\beta)}, \tag{36}$$

or, equivalently,

$$\sqrt{|\gamma + \delta|} * s_{\sigma(\gamma+\delta)} = \sqrt{|\gamma|} * s_{\sigma(\gamma)} \oplus \sqrt{|\delta|} * s_{\sigma(\delta)}. \tag{37}$$

Proof. For $\alpha\beta \geq 0$, the element $s \in \mathbf{S}$ satisfies relation (36), due to (30). We consider other cases. Assume that $0 \leq -\beta \leq \alpha$, and hence $\alpha^2 - \beta^2 = \sigma(\alpha)\alpha^2 + \sigma(\beta)\beta^2 \geq 0$. Using (30) we obtain for $s \in \mathbf{S}$:

$$\alpha * s = \sqrt{(\alpha^2 - \beta^2) + \beta^2} * s = \sqrt{\alpha^2 - \beta^2} * s \oplus \beta * s.$$

Thus for $s \in \mathbf{S}$, $\alpha * s = \sqrt{\alpha^2 - \beta^2} * s \oplus \beta * s$, and hence $\alpha * s \oplus \beta * s_- = \sqrt{\alpha^2 - \beta^2} * s$, showing that (36) is true in this case. Other cases are treated similarly. To obtain (37) substitute in (36) $\gamma = \sigma(\alpha)\alpha^2$, $\delta = \sigma(\beta)\beta^2$. $\square$

Relation (36) can be written as:

$$(\alpha \oplus \beta) * s_{\sigma(\alpha+\beta)} = \alpha * s_{\sigma(\alpha)} \oplus \beta * s_{\sigma(\beta)}, \tag{38}$$

or as:

$$(\alpha \oplus \beta) * s_{\sigma(\alpha \oplus \beta)} = \alpha * s_{\sigma(\alpha)} \oplus \beta * s_{\sigma(\beta)}. \tag{39}$$

Remarks. 1) The above formulae imply that the element s can be factored out in an expression of the form $\alpha * s_{\sigma(\alpha)} \oplus \beta * s_{\sigma(\beta)}$. The same is true for an expression of the form: $\alpha * s_{\sigma(\alpha)} \oplus \beta * s_{\sigma(-\beta)}$ as we can then set $\gamma = -\beta$ and obtain an expression of the previous form (using that $\beta * s_{\sigma(-\beta)} = (-\beta) * s_{\sigma(-\beta)} =$

$\gamma * s_{\sigma(\gamma)}$). 2) Note that $\alpha \oplus \beta$ in the left-hand side of (38) is a shorthand notation (24) for a scalar from the field $\mathbb{R} = (\mathbb{R}, +, \cdot)$, whereas $\alpha * s \oplus \beta * s$ is an element of $\mathbf{S}$.

Using (23) formula (5) can be extended as follows:

$$\sqrt{\gamma}^{\circ} \oplus \sqrt{\delta}^{\circ} = \sqrt{\gamma + \delta}^{\circ}, \; \gamma, \delta \in \mathbb{R}. \tag{40}$$

Using (23) we can write (37) as

$$\sqrt{\alpha + \beta}^{\circ} * s_{\sigma(\alpha+\beta)} = \sqrt{\alpha}^{\circ} * s_{\sigma(\alpha)} \oplus \sqrt{\beta}^{\circ} * s_{\sigma(\beta)}, \tag{41}$$

minding that the sign of the scalar δ in the expression $\delta * s$ does not matter (due to $\delta * s = -\delta * s = |\delta| * s$).

It follows from proposition 11 that if we substitute (30) in Definition 8 by (36), resp. (38), then we obtain an equivalent definition of S-space, that is Definition 8 is equivalent to:

Definition 12 *A system* $(\mathbf{S}, \oplus, \mathbb{R}, *)$*, such that* $(\mathbf{S}, \oplus)$ *is an abelian additive group, and for any* $s, t \in \mathbf{S}$ *and* $\alpha, \beta \in \mathbb{R}$*:*

$$\begin{aligned}
\alpha * (s \oplus t) &= \alpha * s \oplus \alpha * t, & (42)\\
\alpha * (\beta * s) &= (\alpha\beta) * s, & (43)\\
1 * s &= s, & (44)\\
(-1) * s &= s, & (45)\\
(\alpha \oplus \beta) * s_{\sigma(\alpha+\beta)} &= \alpha * s_{\sigma(\alpha)} \oplus \beta * s_{\sigma(\beta)}, & (46)
\end{aligned}$$

is called an S-space over $\mathbb{R}$*.*

We next consider a relation between S-spaces and vector spaces.

5 Relation Between S-spaces and Vector Spaces

5.1 Vector Spaces Induced by S-spaces

Let $(\mathbf{S}, +, \mathbb{R}, *)$ be an S-space over $\mathbb{R}$. Define the operation "$\cdot$": $\mathbb{R} \times \mathbf{S} \longrightarrow \mathbf{S}$ by

$$\alpha \cdot c = \sqrt{|\alpha|} * c_{\sigma(\alpha)} = \sqrt{\alpha}^{\circ} * c_{\sigma(\alpha)}, \; \alpha \in \mathbb{R}, \; c \in \mathbf{S}. \tag{47}$$

Equality (47) is equivalent to:

$$\sigma(\alpha)\alpha^2 \cdot c = \alpha * c_{\sigma(\alpha)}, \ \alpha \in \mathbb{R}, \ c \in \mathbf{S}, \tag{48}$$

the latter meaning: $\alpha^2 \cdot c = \alpha * c$, if $\alpha \geq 0$ and $-\alpha^2 \cdot c = \alpha * c_-$, if $\alpha < 0$.

Proposition 13 *Let* $(\mathbf{S}, +, \mathbb{R}, *)$ *be an S-space over* $\mathbb{R}$. *Then* $(\mathbf{S}, +, \mathbb{R}, \cdot)$, *with "$\cdot$" defined by (47), is a vector space over* $\mathbb{R}$, *that is for every* $\alpha, \beta, \gamma \in \mathbb{R}$, $a, b, c \in \mathbf{S}$*:*

$$\gamma \cdot (a + b) = \gamma \cdot a + \gamma \cdot b, \tag{49}$$
$$\alpha \cdot (\beta \cdot c) = (\alpha\beta) \cdot c, \tag{50}$$
$$1 \cdot a = a, \tag{51}$$
$$(\alpha + \beta) \cdot c = \alpha \cdot c + \beta \cdot c. \tag{52}$$

Proof. To prove (49) substitute $a = c_{\sigma(\gamma)}$, $b = d_{\sigma(\gamma)}$ in $\gamma*(a+b) = \gamma*a+\gamma*b$. We obtain $\gamma * (c_{\sigma(\gamma)} + d_{\sigma(\gamma)}) = \gamma * c_{\sigma(\gamma)} + \gamma * d_{\sigma(\gamma)}$, or $\gamma * (c + d)_{\sigma(\gamma)} = \gamma * c_{\sigma(\gamma)} + \gamma * d_{\sigma(\gamma)}$. This implies $\sigma(\gamma)\gamma^2 \cdot (c + d) = \sigma(\gamma)\gamma^2 \cdot c + \sigma(\gamma)\gamma^2 \cdot d$, for all $c, d \in \mathbf{S}$, $\gamma \in \mathbb{R}$. This is equivalent to (49) if we set $\alpha = \sigma(\gamma)\gamma^2$.

To prove (50) substitute $c = d_{\sigma(\beta)}$ in the relation $\alpha*(\beta*c) = (\alpha\beta)*c$ to obtain $\alpha*(\beta*d_{\sigma(\beta)}) = (\alpha\beta)*d_{\sigma(\beta)}$. Using (48) we have $\alpha*(\sigma(\beta)\beta^2 \cdot d) = (\alpha\beta)*d_{\sigma(\beta)}$, which implies $\alpha * (\beta \cdot d)_{\sigma(\alpha)} = (\alpha\beta) * d_{\sigma(\beta)\sigma(\alpha)} = (\alpha\beta) * d_{\sigma(\beta\alpha)}$, or, using (48), $\sigma(\alpha)\alpha^2(\sigma(\beta)\beta^2 \cdot d) = \sigma(\alpha\beta)(\alpha\beta)^2 \cdot d$, that is (50).

Relation (51): $1 \cdot a = a$ is obviously true. Relation (52): $(\alpha+\beta)\cdot c = \alpha\cdot c+\beta\cdot c$, resp. $(-1) \cdot a + a = 0$, follows from (46) using (48). Indeed, using (25), (47) and (48) we obtain $(\sigma(\alpha)\alpha^2 + \sigma(\beta)\beta^2) \cdot s = \sigma(\alpha)\alpha^2 \cdot s + \sigma(\beta)\beta^2 \cdot s$ which is equivalent to (52).

We thus proved that $(\mathbf{S}, +, \mathbb{R}, \cdot)$ is a vector space. $\square$

Operation (47) is well defined on $\mathbb{R} \times \mathbf{S}$ for any S-space $\mathbf{S}$ over $\mathbb{R}$; it may be called *linear multiplication* in $\mathbf{S}$, whereas, by contrast, the original multiplication "$*$" in $\mathbb{R}\times\mathbf{S}$ may be called *s-multiplication.* The above proposition implies that every S-space $(\mathbf{S}, +, \mathbb{R}, *)$ involves a linear multiplication and hence an *associated vector space* $(\mathbf{S}, +, \mathbb{R}, \cdot)$. Note that the element $(-1)\cdot a = (-1)*a_- = a_-$ is the opposite to a in $\mathbf{S}$, $\mathrm{opp}(a) = a_-$, as we have: $a + (-1) \cdot a = 0$.

The two spaces $(\mathbf{S}, +, \mathbb{R}, *)$ and $(\mathbf{S}, +, \mathbb{R}, \cdot)$ are *equivalent* in the sense that every expression in the first space can be presented in terms of the operations of the second space, and vice versa. Thus we have:

Proposition 14 *Every S-space over* $\mathbb{R}$ *induces via (47) an equivalent vector* *space over* $\mathbb{R}$.

5.2 S-spaces Induced by Vector Spaces

Let $(\mathbf{S},+,\mathbb{R},\cdot)$ be a vector space over $\mathbb{R}$. It can be immediately seen that the system $(\mathbf{S},+,\mathbb{R},*)$, where "$*$" is defined by

$$\alpha * c = \sigma(\alpha)\alpha^2 \cdot c_{\sigma(\alpha)} = \alpha^2 \cdot c \tag{53}$$

is an S-space over $\mathbb{R}$. The two spaces $(\mathbf{S},+,\mathbb{R},*)$ and $(\mathbf{S},+,\mathbb{R},\cdot)$ are *equivalent.* Thus we have:

Proposition 15 *Every vector space over* $\mathbb{R}$ *induces via (53) an equivalent S-space over* $\mathbb{R}$.

Note that the spaces $(\mathbf{S},+,\mathbb{R},*)$ and $(\mathbf{S},+,\mathbb{R},\cdot)$, although equivalent, are generally distinct from each other as they have different operations for multiplication by scalars.

The last two propositions can be summarized as follows: *Every S-space over* $\mathbb{R}$ *generates via (47) an equivalent vector space and, vice versa, every vector space over* $\mathbb{R}$ *induces via (53) an equivalent S-space over* $\mathbb{R}$.

One can make use of both operations for multiplication by scalars simultaneously. The system $(\mathbf{S},+,\mathbb{R},\cdot,*)$ can be viewed either as a vector space over $\mathbb{R}$ endowed with the operation (53) or as an S-space over $\mathbb{R}$ endowed via (47) with the operation "$\cdot$". In $(\mathbf{S},+,\mathbb{R},\cdot,*)$ one has two different notations for the opposite operator. Namely, opposite is denoted in $(\mathbf{S},+,\mathbb{R},\cdot)$ by $\mathrm{opp}(a) = -a$, whereas in $(\mathbf{S},+,\mathbb{R},*)$ one writes $\mathrm{opp}(a) = a_-$.

Using that the spaces $(\mathbf{S},+,\mathbb{R},\cdot)$ and $(\mathbf{S},+,\mathbb{R},*)$ are equivalent we can transfer familiar concepts from the theory of vector spaces to S-spaces. In what follows we briefly outline this idea.

6 Vector space concepts in S-spaces

Assume that $\mathbf{S} = (\mathbf{S},+,\mathbb{R},*)$ is an S-space over $\mathbb{R}$ and $(\mathbf{S},+,\mathbb{R},\cdot)$ is the associated equivalent vector space. From the vector space $(\mathbf{S},+,\mathbb{R},\cdot)$ we can transfer vector space concepts, such as linear combination, linear dependence, basis etc., to the S-space $(\mathbf{S},+,\mathbb{R},*)$. For example, the concept of linear combination obtains the following form.

Let $c^{(1)}, c^{(2)}, ..., c^{(k)}$, $k \geq 1$, be finitely many (not necessarily distinct) elements of $\mathbf{S}$ and let $f = \sum_{i=1}^{k} \gamma_i \cdot c^{(i)} = \gamma_1 \cdot c^{(1)} + \gamma_2 \cdot c^{(2)} + ... + \gamma_k \cdot c^{(k)}$ with $\gamma_1, \gamma_2, ..., \gamma_k \in \mathbb{R}$ be a linear combination of $c^{(1)}, c^{(2)}, ..., c^{(k)}$ in the vector space $(\mathbf{S}, +, \mathbb{R}, \cdot)$. Using (47) we introduce a *linear combination* in the S-space $(\mathbf{S}, +, \mathbb{R}, *)$ by

$$f = \sqrt{|\gamma_1|} * c^{(1)}_{\sigma(\gamma_1)} + \sqrt{|\gamma_2|} * c^{(2)}_{\sigma(\gamma_2)} + ... + \sqrt{|\gamma_k|} * c^{(k)}_{\sigma(\gamma_k)}. \tag{54}$$

Setting $\alpha_i = \sqrt{\gamma_i}^{\circ}$, we can rewrite (54) as

$$f = \alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(2)}_{\sigma(\alpha_2)} + ... + \alpha_k * c^{(k)}_{\sigma(\alpha_k)}. \tag{55}$$

Proposition 16 *The set*

$$span\{c^{(1)}, c^{(2)}, ..., c^{(k)}\} = \{\sum_{i=1}^{k} \alpha_i * c^{(i)}_{\sigma(\alpha_i)} \mid \alpha_i \in \mathbb{R}\} \tag{56}$$

of all linear combinations of $c^{(1)}, c^{(2)}, ..., c^{(k)}$ *is a subspace of* $\mathbf{S}$.

Further, the elements $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathbf{S}$, $k \geq 1$, are *linearly dependent (over* $\mathbb{R}$*)*, if there exists a nontrivial linear combination of $\{c^{(i)}\}$, which is equal to 0, i. e. if there exist a system $\{\alpha_i\}_{i=1}^{k}$ with not all α_i equal to zero, such that

$$\alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(1)}_{\sigma(\alpha_2)} + ... + \alpha_k * c^{(k)}_{\sigma(\alpha_k)} = 0. \tag{57}$$

The elements $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathbf{S}$ are *linearly independent*, if (57) is possible only for the trivial linear combination, such that $\alpha_i = 0$ for all $i = 1, ..., k$.

6.1 Linear Mappings in S-spaces

Let $\mathbf{Q_1} = (\mathbf{Q_1}, +, \mathbb{R}, *)$, $\mathbf{Q_2} = (\mathbf{Q_2}, +, \mathbb{R}, *)$ be two S-spaces

over $\mathbb{R}$ and let $\varphi : \mathbf{Q_1} \longrightarrow \mathbf{Q_2}$ be a linear (homomorphic) mapping, that is:

$$\varphi(x + y) = \varphi(x) + \varphi(y), \tag{58}$$
$$\varphi(\gamma * x) = \gamma * \varphi(x), \quad x, y \in \mathbf{Q_1}, \ \gamma \in \mathbb{R}. \tag{59}$$

It is easy to check that $\varphi(x_-) = (\varphi(x))_-$; more generally any linear mapping satisfies:

$$\varphi(\alpha_1 * x^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * x^{(2)}_{\sigma(\alpha_2)} + ... + \alpha_k * x^{(k)}_{\sigma(\alpha_k)}) = \\ \alpha_1 * \varphi(x^{(1)})_{\sigma(\alpha_1)} + \alpha_2 * \varphi(x^{(2)})_{\sigma(\alpha_2)} + ... + \alpha_k * \varphi(x^{(k)})_{\sigma(\alpha_k)}, \tag{60}$$

where $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{R}$, $x^{(1)}, x^{(2)}, ..., x^{(k)} \in \mathbf{Q_1}$. In particular:

$$\varphi(\alpha * x_\lambda + \beta * y_\mu) = \alpha * \varphi(x)_\lambda + \beta * \varphi(y)_\mu,\ x, y \in \mathbf{Q_1},\ \alpha, \beta \in \mathbb{R}. \tag{61}$$

Obviously condition (61) completely characterizes a linear mapping and can substitute conditions (58) and (59).

Let $(\mathbf{S}, +, \mathbb{R}, *)$ be an S-space, $x^{(1)}, x^{(2)}, ..., x^{(n)} \in \mathbf{S}$ and let $\mathbf{S}^n_{(s)} = (\mathbb{R}^n, \oplus, \mathbb{R}, *)$ be the canonical S-space defined in Example 9. The mapping $\varphi : \mathbf{S}^n_{(s)} \longrightarrow \mathbf{S}$, such that

$$\varphi(\alpha_1, \alpha_2, ..., \alpha_n) = \alpha_1 * x^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * x^{(2)}_{\sigma(\alpha_2)} + ... + \alpha_n * x^{(n)}_{\sigma(\alpha_n)}, \tag{62}$$

is linear. Indeed, relations (58) and (59) are satisfied:

$$\begin{aligned}
\varphi((\alpha_1, \alpha_2, ..., \alpha_n) \oplus (\beta_1, \beta_2, ..., \beta_n)) &= \varphi(\alpha_1 \oplus \beta_1, \alpha_2 \oplus \beta_2, ..., \alpha_n \oplus \beta_n) \\
&= (\alpha_1 \oplus \beta_1) * x^{(1)}_{\sigma(\alpha_1+\beta_1)} + ... + (\alpha_n \oplus \beta_n) * x^{(n)}_{\sigma(\alpha_n+\beta_n)} \\
&= \alpha_1 * x^{(1)}_{\sigma(\alpha_1)} + ... + \alpha_n * x^{(n)}_{\sigma(\alpha_n)} + \beta_1 * x^{(1)}_{\sigma(\beta_1)} + ... \beta_n * x^{(n)}_{\sigma(\beta_n)} \\
&= \varphi(\alpha_1, \alpha_2, ..., \alpha_n) + \varphi(\beta_1, \beta_2, ..., \beta_n); \\
\varphi(\gamma * (\alpha_1, \alpha_2, ..., \alpha_n)) &= \varphi(|\gamma|\alpha_1, |\gamma|\alpha_2, ..., |\gamma|\alpha_n) \\
&= (|\gamma|\alpha_1) * x^{(1)}_{\sigma(|\gamma|\alpha_1)} + ... + (|\gamma|\alpha_n) * x^{(k)}_{\sigma(|\gamma|\alpha_n)}) \\
&= (|\gamma|\alpha_1) * x^{(1)}_{\sigma(\alpha_1)} + ... + (|\gamma|\alpha_n) * x^{(k)}_{\sigma(\alpha_n)}) \\
&= |\gamma| * \varphi(\alpha_1, \alpha_2, ..., \alpha_n) = \gamma * \varphi(\alpha_1, \alpha_2, ..., \alpha_n).
\end{aligned}$$

Denote the n-vector $e^{(i)} = (0, 0, ..., 0, 1, 0, ..., 0)$, where the component 1 is on the i-th place. We consider $e^{(i)}$ as element of ${\mathbf{S_{(s)}}}^n$, where opp$(e^{(i)}) = e^{(i)}_-$ and $\neg e^{(i)} = e^{(i)}$. Relation (62) implies

$$\varphi(e^{(i)}) = \alpha_i * x^{(i)}_{\sigma(\alpha_i)} \mid_{\alpha_i = 1} = x^{(i)},\ i = 1, ..., n. \tag{63}$$

The mapping φ is the only linear mapping from $\mathbf{S}^n$ to $\mathbf{S}$ with the property (63). Indeed, if (63) holds, then by (60),

$$\begin{aligned}
\varphi(\alpha_1, \alpha_2, ..., \alpha_n) &= \varphi(\sum \alpha_i * e^{(i)}_{\sigma(\alpha_i)}) \\
&= \sum \alpha_i * \varphi(e^{(i)})_{\sigma(\alpha_i)} = \sum \alpha_i * x^{(i)}_{\sigma(\alpha_i)}.
\end{aligned}$$

We thus obtain that relation (63): $\varphi(e^{(i)}) = x^{(i)},\ i = 1, ..., n,$ is sufficient to determine the mapping (62). As in the case of vector spaces, every mapping of the set $(e^{(1)}, ..., e^{(n)})$ into $\mathbf{S}$ of the form $\varphi(e^{(i)}) = x^{(i)},\ i = 1, ..., n,$ can be extended to a unique linear mapping of $\mathbf{S}^n_{(s)}$ into $\mathbf{S}$.

We shall now prove that the spaces from Examples 9 and 10 are isomorphic.

Proposition 17 *The spaces* $\mathbf{S}^k_{(s)} = (\mathbb{R}^k, \oplus, \mathbb{R}, *)$, $\mathbf{S}^k_{(v)} = (\mathbb{R}^k, +, \mathbb{R}, \diamond)$ *with operations* $\oplus, *, +, \diamond$, *defined by (31), (32), (34), (35), are isomorphic.*

Proof. For simplicity we shall consider the case $k = 1$, the general case is straightforward; denote $\mathbf{S}_{(s)} = \mathbf{S}^1_{(s)}$, $\mathbf{S}_{(v)} = \mathbf{S}^1_{(v)}$.

Consider the operator $\phi : \mathbf{S}_{(s)} \longrightarrow \mathbf{S}_{(v)}$, defined by

$$\phi(x) = \sigma(x)x^2.$$

Note that $\phi(\sqrt{y}^{\circ}) = y$ for $y \in \mathbb{R}$. We shall show that the operator ϕ is linear. Indeed, we have:

$$\begin{aligned} \phi(x \oplus y) &= \phi(\sqrt{\sigma(x)x^2 + \sigma(y)y^2}) = \sigma(x)x^2 + \sigma(y)y^2 = \phi(x) + \phi(y); \\ \phi(\gamma * x) &= \phi(|\gamma| \cdot x) = \gamma^2 \cdot (\sigma(x)x^2) = \gamma \diamond (\sigma(x)x^2) = \gamma \diamond \phi(x), \end{aligned}$$

which proves the theorem. $\square$

Remark. Alternatively, one can prove that the operator $\psi : \mathbf{S}_{(v)} \longrightarrow \mathbf{S}_{(s)}$, defined by

$$\psi(u) = \sqrt{u}^{\circ},$$

is linear. Indeed, using (40) we have:

$$\begin{aligned} \psi(u + v) &= \sqrt{u + v}^{\circ} = \sqrt{u}^{\circ} \oplus \sqrt{v}^{\circ} = \psi(u) \oplus \psi(v); \\ \psi(\delta \diamond u) &= \psi(\delta^2 \cdot u) = \sqrt{\delta^2 \cdot u}^{\circ} = |\delta| \cdot \sqrt{u}^{\circ} = \delta * \sqrt{u}^{\circ} = \delta * \psi(u). \end{aligned}$$

As the two spaces $\mathbf{S}^k_{(s)}$ and $\mathbf{S}^k_{(v)}$ are isomorphic, we may write $\mathbf{S}^k$ meaning any one of the two spaces.

7 Basis in an S-space

Let $\mathbf{S}$ be a S-space over $\mathbb{R}$. The set $\{c^{(i)}\}_{i=1}^k$, $c^{(i)} \in \mathbf{S}$, $k \geq 1$, is a *basis* of $\mathbf{S}$, if $c^{(i)}$ are linearly independent and $\mathbf{S} = \text{span}\{c^{(i)}\}_{i=1}^k$.

Proposition 18 *A set $\{c^{(i)}\}_{i=1}^k$, $c^{(i)} \in \mathbf{S}$, $k \geq 1$, is a basis of $\mathbf{S}$, iff every $f \in \mathbf{S}$ can be presented in the form (55) in a unique way (i. e. with unique scalars α_i).*

Let $\mathbf{S}$ be a S-space over $\mathbb{R}$ and $\{c^{(i)}\}_{i=1}^k$ be a basis of $\mathbf{S}$. Assume that $a = \sum_{i=1}^k \alpha_i * c^{(i)}_{\sigma(\alpha_i)}$, $b = \sum_{i=1}^k \beta_i * c^{(i)}_{\sigma(\beta_i)}$ are two elements of $\mathbf{S}$. Their sum is

$$a + b = \sum_{i=1}^k \alpha_i * c^{(i)}_{\sigma(\alpha_i)} + \sum_{i=1}^k \beta_i * c^{(i)}_{\sigma(\beta_i)} = \sum_{i=1}^k (\alpha_i \oplus \beta_i) * c^{(i)}_{\sigma(\alpha_i+\beta_i)}. \tag{64}$$

Multiplication by scalars is given by

$$\gamma * a = \sum_{i=1}^k |\gamma|\alpha_i * c^{(i)}_{\sigma(\alpha_i)} = \sum_{i=1}^k |\gamma|\alpha_i * c^{(i)}_{\sigma(|\gamma|\alpha_i)}. \tag{65}$$

To every $a = \sum_{i=1}^k \alpha_i * c^{(i)}_{\sigma(\alpha_i)} \in \mathbf{S}$ we associate the vector $(\alpha_1, \alpha_2, ..., \alpha_k)$. Then, minding formulae (64), (65), we define addition and multiplication by scalars by means of (34), (35), arriving thus to the canonic S-space $\mathbf{S}^k_{(s)} = (\mathbb{R}^k, \oplus, , \mathbb{R}, *)$ considered in example 9.

As we know, negation in $\mathbf{S}$ is same as identity. Opposite in $\mathbf{S}$ is $a_- = \mathrm{opp}(a) = \sum_{i=1}^k \alpha_i * c^{(i)}_{-\sigma(\alpha_i)} = \sum_{i=1}^k (-\alpha_i) * c^{(i)}_{\sigma(-\alpha_i)}$, or, in terms of $\mathbf{S}^k_{(s)} = (\mathbb{R}^k, \oplus, \mathbb{R}, *)$ we obtain (33).

Theorem 19 *Any S-space over $\mathbb{R}$, with a basis of k elements, is isomorphic to $\mathbf{S}^k$.*

Proof. Let $\mathbf{S}$ be a S-space spanned over a finite basis $s^{(1)}, s^{(2)}, ..., s^{(k)}$. The linear mapping $\varphi : \mathbf{S}^k \longrightarrow \mathbf{S}$, $\mathbf{S}^k = (\mathbb{R}^k, \oplus, \mathbb{R}, *)$, defined by

$$\varphi(\alpha_1, \alpha_2, ..., \alpha_k) = \alpha_1 * s^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * s^{(2)}_{\sigma(\alpha_2)} + ... + \alpha_k * s^{(k)}_{\sigma(\alpha_k)},$$

is a bijection. Hence φ is an isomorphism. □

Let $\mathbf{S}$ be a S-space spanned over a finite basis $s^{(1)}, s^{(2)}, ..., s^{(k)}$. As in the vector case, the number k (which does not change with the particular basis) will be called *dimension* of $\mathbf{S}$.

Stochastic numbers $(m; s)$ can be considered as elements of a direct sum $V \oplus \mathbf{S}$ of a vector space V and a S-space $\mathbf{S}$. Assume that V and $\mathbf{S}$ have finite bases of dimension k. Namely, let $V = \mathbf{V}^k$ be a k-dimensional vector space with a basis $(v^{(1)}, ..., v^{(k)})$ and let $\mathbf{S} = \mathbf{S^k}$ be a k-dimensional S-space having a basis $(s^{(1)}, ..., s^{(k)})$. Then we say that $(v^{(1)}, ..., v^{(k)}; s^{(1)}, ..., s^{(k)})$ is a basis of the *k-dimensional* space $\mathbf{V}^k \oplus \mathbf{S}^k$. Such a setting allows us to consider numerical problems involving vectors and matrices, wherein the numeric variables have been substituted by stochastic ones. The following example presents a linear system $Ax = b$, such that the right-hand side vector b consists of stochastic numbers. Then the solution vector x also consists of stochastic numbers, and, respectively, all arithmetic operations (additions and multiplications by scalars) in the expression Ax are interpreted in the spirit of this work (therefore we write $A \ {}_{s}\!* \ x$ instead of Ax).

Example 20 *An algebraic problem. Assume that* $A = (\alpha_{ij})_{i,j=1}^n$, $\alpha_{ij} \in \mathbb{R}$ *is a real* $n \times n$*-matrix, and* $b = (b'; b'')$ *is an* n*-vector of (generalized) stochastic numbers, such that* $b', b'' \in \mathbb{R}^n$*. We look for a (generalized) stochastic vector* $x = (x'; x'')$, $x', x'' \in \mathbb{R}^n$*, that is an* n*-vector of stochastic numbers, such that* $A \ {}_{s}\!* \ x = b$*.*

Solution. *Clearly, the system* $A \ {}_{s}\!* \ x = b$ *reduces to a linear system* $Ax' = b'$ *for the vector* x' *of mean values and a system* $A * x'' = b''$ *for the "vector"* x'' *of standard deviations.*

The elements of the vector $A * x''$ *are* $c_i = \alpha_{i1} * x_1'' \oplus ... \oplus \alpha_{in} * x_n''$, $i = 1, ..., n$.

Setting $sign(x_i'')(x_i'')^2 = y_i$, $sign(b_i'')(b_i'')^2 = c_i$, *we obtain a linear* $n \times n$ *system* $Dy = c$ *for* $y = (y_i)$, *where* $D = (\alpha_{ij}^2)$. *If* D *is nonsingular we can solve the system* $Dy = c$ *for the vector* y, *and then obtain the standard deviation vector* x *by means of* $x_i = \sqrt{y_i}^{\circ} = sign(y_i)\sqrt{|y_i|}$. *Thus for the solution of the original problem it is necessary and sufficient that both matrices* $A = (\alpha_{ij})$ *and* $D = (\alpha_{ij}^2)$ *are nonsingular.*

8 Conclusion

In this work we outline an algebraic theory of stochastic numbers related to the operations addition and multiplication by scalar. In the development of this theory we follow similar studies related to convex bodies and intervals. [1], [2], [6], [9].

A fundamental relation for the present theory is the distributivity property (10) for standard deviations corresponding to the second distributive law in a vector space.

The above theoretic study of the properties of stochastic numbers allow us to obtain rigorous abstract definition of stochastic numbers with respect to the operations addition and multiplication by scalars. Our theory also allows us to solve algebraic problems with stochastic numbers (like the problem given at the end of the last section). This gives us a possibility to compare algebraically obtained results with practical applications of stochastic numbers, such as the ones provided by the CESTAC method [3]. Such comparisons will give additional information related to the stochastic behaviour of random roundings in the course of numerical computations.

Future work. The present work is a step towards the formalization of the concept of stochastic numbers in the manner this has been done with real numbers. A further step would concern properties of stochastic numbers related to (inner) multiplication and certain order relations.

Acknowledgments. This work was partially supported by Bulgarian NSF grants I-903/99, MM-1104/01, and Swiss NSF Co-operation Agreement No. 7IP 65642 (Program SCOPES).

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