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On the Quasivector Space of Zonotopes in the Plane

Svetoslav Markov*

Institute of Mathematics & Informatics, BAS, "Acad. G. Bonchev" st., bl. 8, 1113 Sofia, Bulgaria

In this work we consider centred zonogons represented implicitly as Minkowski sums of centred segments. Using this representation the order relation inclusion (containment) of zonogons has been studied.

1 Introduction

Zonotopes are abelian cancellative monoids with respect to addition (in Minkowski sense). With respect to multiplication by scalars they satisfy the axioms of a linear space with one exception: the second distributive law is weakened up to a so-called quasidistributive law, stating that distributivity holds only for equally signed scalars [2]. These spaces naturally involve a partial order relation — inclusion, which is isotone with respect to both addition and multiplication by scalars. In this work we discuss the inclusion relation between centred zonogons (centred zonotopes in the plane). Our purpose is to use the implicit presentation of zonotopes as Minkowski sums of linear segments, as for many applications such a presentation provides fast and effective computations [1]. Within such frames we formulate some conditions for inclusion (containment) of centred zonotopes in the plane that can be used to construct a numerical algorithm for testing inclusion.

A centrally symmetric convex body centred at the origin is called *centred convex body* (cf. [4], p. 383). In what follows we restrict ourselves to zonogons (2D-zonotopes), that is zonotopes in the Euclidean plane $\mathbb{E}^2$ with a fixed coordinate system Oxy. Zonogons have several different presentations. We make use of a presentation based on the Minkowski sum of segments. We refer to this presentation as "implicit" in contrast to the presentation by vertices which is sometimes denoted as "explicit".

2 Centred Zonogons: the Implicit Form

Every unit vector $e = (\cos\varphi, \sin\varphi) \in \mathbb{E}^2,\ \varphi \in [0, \pi)$, defines a centred segment $\tilde{e}$ with endpoints $-e$ and e, symbolically $\tilde{e} = \mathrm{conv}\{-e, e\} = \{\lambda e \mid \lambda \in [-1, 1]\}$, where "conv" means "convex hull" [4]. In the sequel Ov denotes the line passing through the origin O and the point $v \in \mathbb{E}^2$ and $\tilde{v}$ (or $\tilde{v}$) denotes the centred segment on the line Ov comprising all points between $-v$ and v, that is $\tilde{v} = \mathrm{conv}\{-v, v\}$. Note that v is a vector in $\mathbb{E}^2$, whereas $\tilde{v}$ is a centred linear segment.

Assume that we are given a mesh of $k \geq 2$ numbers (angles) $\varphi_i, i = 1, ..., k$, in the interval $[0, \pi)$, such that $0 \leq \varphi_1 < \varphi_2 < ... < \varphi_k < \pi$. Such a system $\{\varphi_i\}_{i=1}^k$ is called *regular*; in the sequel we assume $\varphi_1 = 0$ which is no loss of generality. Every φ_i defines a unit vector $e^{(i)} = (\cos\varphi_i, \sin\varphi_i)$, respectively a centred unit line segment: $\tilde{e}^{(i)} = \mathrm{conv}\{-e^{(i)}, e^{(i)}\}$. A system of unit vectors $e = \{e^{(1)}, e^{(2)}, ..., e^{(k)}\}$, resp. unit line segments $\tilde{e} = \{\tilde{e}^{(1)}, \tilde{e}^{(2)}, ..., \tilde{e}^{(k)}\}$, induced by a regular system of angles is also called *regular*. The elements of the systems e, resp. $\tilde{e}$, are cyclically anticlockwise ordered; the point $e^{(1)}$ lies on the Ox axis of the plane coordinate system Oxy.

For $\alpha_i \geq 0, \quad i = 1, ..., k$, the vectors $a_i = \alpha_i e^{(i)} = (\alpha_i \cos\varphi_i, \alpha_i \sin\varphi_i)$ induce centred segments $\tilde{a}_i = (\alpha_i e^{(i)})\tilde{}$ each of half-length α_i. The *positive combination* of unit centred segments

$$a = \sum_{i=1}^{k} \tilde{a}_i = \sum_{i=1}^{k} (\alpha_i e^{(i)})\tilde{} = \{\sum_{i=1}^{k} \gamma_i e^{(i)} \mid \gamma_i \in [-\alpha_i, \alpha_i]\},\ \alpha_i \geq 0, \tag{1}$$

is a centred zonogon (centred *zonotope* in the plane $\mathbb{E}^2$); the sum of the segments $\tilde{a}_i$ in (1) is understood in Minkowski sense (as vector sum). In particular, the Minkowski sum $z = \tilde{u} + \tilde{v}$ of two centred line segments $\tilde{u}, \tilde{v}$ induced by the points u, v, is a (centred) parallelogram in the plane.

The zonogon (1) has $2k$ vertices: $v^{(1)}, v^{(2)}, ..., v^{(k)}, -v^{(1)}, -v^{(2)}, ..., -v^{(k)}$, which can be computed straightforward, e. g. [4]: $v^{(i)} = -\alpha_1 e^{(1)} - ... - \alpha_{i-1} e^{(i-1)} + \alpha_i e^{(i)} + ... + \alpha_k e^{(k)},\ i = 1, ..., k$.

In the sequel centred zonogons will be presented in the form (1) with a fixed regular system $e = \{\tilde{e}^{(i)}\}$ further referred as basic. Hence, a centred zonogon $a = \sum_{i=1}^{k} (\alpha_i e^{(i)})\tilde{}$ can be identified with its "coordinates" $\alpha_k i$, we thus write $a = (\alpha_1, ..., \alpha_k)$.

The use of a presentation (1) of zonogons is justified by the fact that we are able to approximate any centred zonogon over arbitrary regular system of unit segments by means of zonogons of the class (1) using a fixed basic system $\{\tilde{e}^{(i)}\}$ of centred unit segments. An algorithm for outward approximation producing tight enclosures has been proposed in [2].

* e-mail: smarkov@bio.bas.bg, Phone: +00 3592 979 3704, Fax: +00 3592 971 3649

3 Testing Inclusion Between Two Centred Zonogons in Implicit Form

We know that $\tilde{a}_i \subseteq \tilde{b}_i, i = 1, ..., k \implies a \subseteq b$, but the inverse $a \subseteq b \implies \tilde{a}_i \subseteq \tilde{b}_i$ may not be true as the example with $k = 3$, $a = (0, 1, 0)$ and $b = (1, 0, 1)$ shows. From this example we see that the condition $\tilde{a}_i \subseteq \tilde{b}_i$ may not be satisfied for all $i = 1, ..., k$ and nevertheless $a \subseteq b$ may hold. In our case: $(0, 1, 0) \subseteq (1, 0, 1)$ but the condition $\tilde{a}_2 \subseteq \tilde{b}_2$ does not hold. However, as the "neighbouring conditions" $\tilde{a}_1 \subseteq \tilde{b}_1$, $\tilde{a}_3 \subseteq \tilde{b}_3$ are satisfied "with excess", we have $a \subseteq b$.

Now let us take a regular system of unit segments and consider two centred zonogons $a = (\alpha_1, ..., \alpha_k), b = (\beta_1, ..., \beta_k)$. Assume that the conditions $\tilde{a}_i \subseteq \tilde{b}_i$ hold for all $i = 1, ..., k$ except for some indexes $j, 1 < j < k$, for which $a_j \supset b_j$. For such j we may assume w. l. o. g. that $\beta_j = 0$ by redefining α_j as $\alpha_j - \beta_j$. For example, the inclusion $(0, 2, 0) \subseteq (1, 1, 1)$ is equivalent to $(0, 1, 0) \subseteq (1, 0, 1)$. By a similar reason we can assume $\alpha_i = 0$ for $i = 1, ..., k,\ i \neq j$. Such a redefinition is possible due to the following

Proposition 3.1 *Assume that* $(\alpha_1, ..., \alpha_k)$, $(\beta_1, ..., \beta_k)$ *are two implicitly given zonogons. We have* $(\alpha_1, ..., \alpha_k) \subseteq (\beta_1, ..., \beta_k)$ $\iff (\alpha'_1, ..., \alpha'_k) \subseteq (\beta'_1, ..., \beta'_k)$, *wherein*

$$\alpha'_i = \begin{cases} 0, & \text{if} \quad \alpha_i \leq \beta_i, \\ \alpha_i - \beta_i, & \text{if} \quad \alpha_i > \beta_i; \end{cases} \qquad \beta'_i = \begin{cases} 0, & \text{if} \quad \alpha_i \geq \beta_i, \\ \beta_i - \alpha_i, & \text{if} \quad \alpha_i < \beta_i. \end{cases}$$

Proposition 3.1 suggests in the simplest nontrivial case the necessity of checking the inclusion $(0, 0, ..., 0, \alpha_j, 0, ..., 0) \subseteq (\beta_1, \beta_2, ..., \beta_{j-1}, 0, \beta_{j+1}, ..., \beta_k)$ (with the resp. modification for the case $j = 1$ or $j = k$). We shall consider this case in some detail. Denote the angle between the lines $Oe^{(j)}$ and $Oe^{(j-1)}$ by φ and the angle between the lines $Oe^{(j)}$ and $Oe^{(j+1)}$ by ψ. We first wish to check the inclusion $\tilde{a}_j \subseteq \tilde{b}_{j-1} + \tilde{b}_{j+1}$, which is clearly equivalent to $a = (0, 0, ..., 0, \alpha_j, 0, ..., 0) \subseteq (0, 0, ..., 0, \beta_{j-1}, 0, \beta_{j+1}, 0, ..., 0) = b$.

Proposition 3.2 *We have* $(0, 0, ..., 0, \alpha_j, 0, ..., 0) \subseteq (0, 0, ..., 0, \beta_{j-1}, 0, \beta_{j+1}, 0, ..., 0)$ *if and only if* $\alpha_j \leq \min\{\beta^{(j)}_{j-1}, \beta^{(j)}_{j+1}\}$, *wherein* $\beta^{(j)}_{j-1} = \beta_{j-1} \cos\varphi(\tan\varphi / \tan\psi + 1)$, $\beta^{(j)}_{j+1} = \beta_{j+1} \cos\psi(\tan\psi / \tan\varphi + 1)$.

Note that in the practically important case $\varphi = \psi$ we have the simpler expressions $\beta^{(j)}_{j-1} = 2\beta_{j-1} \cos\varphi$, $\beta^{(j)}_{j+1} = 2\beta_{j+1} \cos\psi$.

We see that the inclusion $(0, 0, ..., 0, \alpha_j, 0, ..., 0) \subseteq (0, 0, ..., 0, \beta_{j-1}, 0, \beta_{j+1}, 0, ..., 0)$ holds true if both values $\beta^{(j)}_{j-1}$, $\beta^{(j)}_{j+1}$ exceed the value of α_j. If these values are not sufficiently large, then, using a similar technique, we should check further whether the next neighbouring values β_{j-2}, β_{j+2} are large enough and so on. An algorithm for testing inclusion of centred zonogons based on prpositions of the above type has been formulated and implemented in CAS Mathematica.

Concluding remarks. Zonotopes are a suitable tool for bounding regions of uncertainty, enclosing medical images, objects in robotics and technical sciences etc. [1]. To simplify computations, it is desirable to consider zonotopes from a finite-parametric family, with a fixed number of parameters; such a natural family using regular basic vectors has been used in the paper. As addition and multiplication by scalar within such a family is straightforward, we concentrate on the inclusion relation. Our study of the presentation and algebraic computation with zonotopes has been guided by the theory of quasivector spaces [2]. As every quasivector space is a direct sum of a linear subspace and a symmetric quasivector subspace we concentrate on the space of centrally symmetric zonotopes centred at the origin (centred zonotopes) which can be presented as Minkowski sums of centred segments. The theory of finite quasivector spaces can be effectively applied to zonotopes in implicit form. Within such frames we formulate conditions for inclusion (containment) of zonotopes in the plane from a certain class that can be easily implemented in a numerical algorithm.

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