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On the Foundations of Interval Arithmetic

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0 Introduction

We consider a special class of abelian semigroups $(S, +)$, such that for every ordered couple (A, B), $A, B \in S$, either $A + X = B$ or $B + Y = A$ is uniquely solvable. We assume that there is an operator "negation" ("reciprocal" in the multiplicative case) in $(S, +)$ having certain properties abstracted from the interval arithmetic operator "$-$" (resp. "$/$") in $I(\mathbb{R})$. The operator negation generates the operations subtraction, inner addition and inner subtraction. We show that negation plays a special role in the embedding of the semigroup S in the set $D = S \otimes \{+, -\}$ of elements of the form $(A; \pm)$, $A \in S$. We establish an isomorphism between D and the group $\mathcal{G} = S^2/E$ with E : $(A, B) \sim (U, V) \Longleftrightarrow A + V = B + U$. In $\mathcal{G}$ the inverse element is $\mathrm{opp}(A, B) = (B, A)$ and in D we have the natural operator $\mathrm{dual}(A; \alpha) = (A; -\alpha)$, $\alpha = \pm$. We show that opp and dual are related in $\mathcal{G}$ by $\mathrm{dual}(A, B) = -\mathrm{opp}(A, B) = -(B, A) = (-B, -A)$, resp. in D by $\mathrm{opp}(A; \alpha) = -\mathrm{dual}(A; \alpha) = -(A; -\alpha) = (-A; -\alpha)$. This allows a transition of results between $\mathcal{G}$ and D and a "projection" of results from D on S. Analogies of the results in the groups $\mathcal{G}$ and D can be found in S, which leads to an enrichment of S with operators, operations and corresponding relations. For instance, an associative-like relation is obtained in S by translating the associative law in the group D. The results obtained are illustrated on interval arithmetic systems.

1 Semigroups with Negation

Let $(S, +)$ be a commutative (abelian) semigroup, which satisfies the following property, further refered as t-property: *For every pair (A, B), $A, B \in S$, at least one of the equations $A + X = B$, $B + Y = A$ has a solution, which is thereby unique.* It is easily seen that if $(S, +)$, satisfies the t-property, then it obeys a cancellation law.

Proposition 1.1. *The t-property is equivalent to the following trichotomy condition (T): For (A, B), $A, B \in S$, exactly one of the following three cases holds true:*

T1) Equation $A + X = B$ has a unique solution, whereas $B + Y = A$ has no solution;

T2) Equation $B + Y = A$ has a unique solution, equation $A + X = B$ has no solution;

T3) Both equations $A + X = B$ and $B + Y = A$ have unique solutions.

If (A, B) satisfies T1), then (B, A) satisfies T2) and vice versa. The elements X, Y in case T3) are called *degenerate*, i. e. $X \in S$ is degenerate if there exist $A, B \in S$, satisfying T3). The set of all degenerate elements of S is denoted by S_0.

Proposition 1.2. *Every semigroup $(S, +)$, satisfying a t-property, possesses a null element, i. e. $A + X = A$ has a unique solution $X = 0$ not depending on $A \in S$.*

The null element is degenerate, $0 \in S_0$; $A + A = A$ is equivalent to $A = 0$ and $A + 0 = B$ is equivalent to $A = B$. Since $0 + 0 = 0$, $(\{0\}, +)$ is a (trivial) group, which is a subgroup of $(S, +)$. S_0 is a group, which is a subgroup of S. The inverse element to $A \in S_0$ is denoted, by $-A$, so that $A + (-A) = 0$. A semigroup S containing only the trivial subgroup $S_0 = \{0\}$ is called simple. A semigroup $(S, +)$ is called *proper* if $(S, +)$ is not a group in its own right. A semigroup $(S, +)$ is called *nonsimple* if there exists at least one element $P \in S$, $P \neq 0$, such that $P + X = 0$ has a unique solution. A proper, nonsimple semigroup $(S, +)$, satisfying a t-property, will be briefly called a *pnt-semigroup.* For a pnt-semigroup $(S, +)$ with a subgroup S_0 we denote $S_+ = S \setminus S_0$. By definition, $S_+ \neq \emptyset$.

Definition 1.1. *The trichotomy (T) generates an order relation $\leq_\omega$ in S:*

T1) $A <_\omega B$ means that B can be presented as a sum of A and an element $X \in S_+$, symbolically, $B = A + X$, $X \in S_+$, whereas A cannot be presented as $B + Y$, $Y \in S_+$; the unique X, satisfying $A + X = B$ is denoted by $X = B \dot{-} A \in S_+$.

T2) $A >_\omega B$ means that A can be presented as a sum of B and an element $Y \in S_+$, symbolically, $A = B + Y$, $Y \in S_+$, whereas $B = A + X$, $X \in S_+$ is not true; the unique solution Y of $B + Y = A$ is denoted by $Y = A \dot{-} B \in S_+$.

T3) $A =_\omega B$ means that both: i) B can be presented as a sum of A and an element $X \in S_0$, that is $A + X = B$, and ii) A can be presented as a sum of B and some element $Y \in S_0$, that is $B + Y = A$; we have $Y = -X$.

The relation $A =_\omega B$ is equivalent to $A = B + P$, $P \in S_0$, and is an equivalence relation in S. We have $A \in S_0 \Longleftrightarrow A =_\omega 0$. The relation "$=_\omega$" generates equivalence classes, which are cosets of S of the form $A + S_0 = \{A + P \mid P \in S_0\}$, $A \in S$; $A =_\omega B$ means that A and B belong to same cosets: $A =_\omega B \Longleftrightarrow A \in S_0 + B \Longleftrightarrow B \in S_0 + A$. In a pnt-semigroup $(S, +)$ with a subgroup S_0 we define negation as follows:

Definition 1.2. *Let S be a pnt-semigroup. An operator $N : S \longrightarrow S$ with the properties: i) $N(A) = 0 \Longleftrightarrow A = 0$, $A \in S$; ii) $N(N(A)) = A$, $A \in S$; iii) $N(P) = -P$, $P \in S_0$; iv) $N(A + B) = N(A) + N(B)$, $A, B \in S$, is called negation (in S).*

Proposition 1.3. *Let $(S, +)$ be a pnt-semigroup and $N : S \longrightarrow S$ be negation. Then i) N is an automorphism and $N^{-1} = N$; ii) $N(A) =_\omega A$ for $A \in S$; iii) For $A \in S$, $N(A + S_0) = A + S_0$, that is the image of a coset under N is the same coset. In particular, N maps S_0, S_+, and S resp. into S_0, S_+ and S.*

Proposition 1.4. *Let N be a negation in the pnt-semigroup $(S,+)$ and let for every $A \in S$ there is a $Q \in S_0$, such that $N(A+Q) = A+Q$. Then N is unique in S.*

Proof. We first show that if for $A \in S$ there is a $Q = Q(A) \in S_0$, with $N(A+Q) = A+Q$, then Q is unique (for this A). Assume that for $Q^* \in S_0, Q^* \neq Q$, we have $N(A+Q^*) = A+Q^*$. Then from $N(A) = A+Q+Q$ and $N(A) = A+Q^*+Q^*$ follows that $Q+Q = Q^*+Q^*$ that is $Q = Q^*$. We next show that the element $A+Q = A+Q(A)$ is fixed for all A belonging to one and the same coset. Indeed let $B = A+P$ for $P \in S_0$ and $Q_1 = Q(B)$. Replacing $B = A+P$ in $N(B+Q_1) = B+Q_1$ we obtain $N(A+P+Q_1) = A+P+Q_1$, which, compared to $N(A+Q) = A+Q$, gives $P+Q_1 = Q$. Hence, $A+Q = A+P+Q_1 = B+Q_1$. □

The class of all pnt-semigroups possessing a unique operator negation will be further denoted by $\mathcal{I}$. The unique negation in $S \in \mathcal{I}$ will be denoted by $N(A) = -A$, and we shall write $S = (S, S_0, +, -) \in \mathcal{I}$. Since $S \in \mathcal{I}$ is not a group and therefore has no inverse elements (besides the elements of the subgroup S_0, for which inverse and negation coincide) this notation causes no confusion and is consistent with the familiar notation "$-$" in interval arithmetic. We introduce in S the operation $A - B = A + (-B)$, $A, B \in S$. The operations $A+B$, $A-B$ are sometimes called outer. The inner addition is defined by

$$A +^- B = \begin{cases} A \dot{-} (-B), & A \geq_\omega B, \\ B \dot{-} (-A), & A \leq_\omega B. \end{cases} \tag{1}$$

For $A =_\omega B$ the expressions in the right-hand side of (1) coincide. The addition $+^-$ is commutative; the additions $+, +^-$ are (conditionally) associative in the sense of Proposition 2.2 below. The inner subtraction is defined by $A -^- B = A +^- (-B) = \{A \dot{-} B, \text{ if } A \geq_\omega B;\ -(\dot{-} A), \text{ if } A \leq_\omega B\}$; it satisfies $A -^- A = 0$.

2 Algebraic Completion of the Semigroup

It is well known that every abelian semigroup $(S,+)$ can be isomorphically embedded in an abelian group $(\mathcal{G},+)$ using the following scheme. Denote by $(\mathcal{G},+)$ the abelian group generated by $(S,+)$ such that $\mathcal{G} = S^2/E$, where $S^2 = \{\langle A,B\rangle \mid A,B \in S\}$, and $E : \langle A,B\rangle \sim \langle P,Q\rangle$, iff $A+Q = B+P$. The elements of $\mathcal{G}$ are the equivalence classes of S^2 w. r. t. E, and are denoted by (A,B), $A,B \in S$. Obviously, $(A,B) = (A+X, B+X)$ for any $X \in S$. In particular, $(A,0) = (A+X,X)$, $(0,B) = (X,B+X)$ and $(0,0) = (X,X)$ for any $X \in S$. The group operation $+$ in $\mathcal{G}$ is defined by $(A,B)+(P,Q) = (A+P, B+Q)$, $A,B,P,Q \in S$. The null element in $(\mathcal{G},+)$ is $(A,A) = (0,0)$ and the inverse of (A,B) is (B,A). We denote the inverse elements in $(\mathcal{G},+)$ by $\operatorname{opp}(A,B) = (B,A)$ (and not by "$-$" in order to avoid confusion with negation). The trichotomy (T) subdivides $\mathcal{G}$ into the subsets $\mathcal{G}_+$, $\mathcal{G}_-$, $\mathcal{G}_0$, of resp.

proper, improper and degenerate elements:

$$\begin{aligned}
\mathcal{G}_+ &= \{(A,B) \mid A >_\omega B\} = \{(B+X,B) \mid X \in S_+\} = \{(X,0) \mid X \in S_+\},\\
\mathcal{G}_- &= \{(A,B) \mid A <_\omega B\} = \{(A,A+Y) \mid Y \in S_+\} = \{(0,Y) \mid Y \in S_+\},\\
\mathcal{G}_0 &= \{(A,B) \mid A =_\omega B\} = \{(X,0) \mid X \in S_0\} = \{(0,Y) \mid Y \in S_0\},
\end{aligned}$$

where $S_+ = S \setminus S_0$, $X = A \dot{-} B$, $Y = B \dot{-} A$. We have $\mathcal{G} = \mathcal{G}_+ \bigcup \mathcal{G}_0 \bigcup \mathcal{G}_-$ and $\mathcal{G}_0$ is a subgroup of $\mathcal{G}$. The semigroup $(S,+)$ is isomorphically embedded in $(\mathcal{G},+)$ under $\varphi : S \longrightarrow \mathcal{G}_{+,0}$, where $\mathcal{G}_{+,0} = \mathcal{G}_+ \bigcup \mathcal{G}_0 = \{(X,0) \mid X \in S\}$ with $\varphi(A) = (A,0)$, $A \in S$. For $P \in S_0$ we have $\varphi(P) = (P,0) = (0,-P)$, since $P+(-P) = 0$ (S_0 is a group). The image of S under φ is $\varphi(S) = \mathcal{G}_{+,0}$; we also have $\varphi(S_+) = \mathcal{G}_+$ and $\varphi(S_0) = \mathcal{G}_0$. We may use an unique representation of the elements of the group $(\mathcal{G},+)$, generated by $(S,+) \in \mathcal{I}$, by writing the elements of $\mathcal{G}_{+,0} = \mathcal{G}_0 \bigcup \mathcal{G}_+$ in the form $(A,0)$ with $A \in S$, and the elements of $\mathcal{G}_-$ in the form $(0,B)$ with $B \in S_+$, that is

$$\mathcal{G}_0 = \{(P,0) \mid P \in S_0\},\ \mathcal{G}_+ = \{(A,0) \mid A \in S_+\},\ \mathcal{G}_- = \{(0,B) \mid B \in S_+\}. \tag{2}$$

The group $\mathcal{G}_0$ subdivides $\mathcal{G}$ into disjoint cosets of the form $(A,B)+\mathcal{G}_0$. For $A \in S_+$ the sets $(A,0)+\mathcal{G}_0 = \{(A+P,0) \mid P \in S_0\} \subset \mathcal{G}_+$, $(0,A)+\mathcal{G}_0 = \{(0,A+Q) \mid Q \in S_0\} \subset \mathcal{G}_-$ are cosets involving proper, resp. improper elements. To add two elements of $\mathcal{G}$, one of which belongs to $\mathcal{G}_-$ and the other to $\mathcal{G}_+$, we may use the following expression which gives the sum in the unique form (2):

$$(A,0)+(0,B) = (A,B) = \begin{cases} (A \dot{-} B, 0), & A \geq_\omega B,\\ (0, B \dot{-} A), & A \leq_\omega B. \end{cases} \tag{3}$$

Embedding using directed elements. Let $(S,S_0,+,-) \in \mathcal{I}$. An ordered couple of the form $(A;\alpha)$, $A \in S$, $\alpha \in \{+,-\}$, is called a *directed element generated by* S. We define the following sets of directed elements generated by S:

$$\begin{gathered}
D_+ = \{(A;+) \mid A \in S_+\},\ D_- = \{(A;-) \mid A \in S_+\},\ D_0 = \{(A;+) \mid A \in S_0\};\\
D_{+,0} = \{(A;+) \mid A \in S\} = D_+ \bigcup D_0,\ D = D_{+,0} \bigcup D_- = D_+ \bigcup D_- \bigcup D_0.
\end{gathered}$$

In the above definition of D we do not use elements of the form $(A;-)$, with $A \in S_0$. For convenience we set $(A;-) = (A;+)$ for $A \in S_0$; then $D = S \otimes \Lambda$, $\Lambda = \{+,-\}$. In Λ we introduce a "multiplication": $++ = -- = +, +- = -+ = -$. For $A \in S_+$, the element $(A;+) \in D_+$ is called proper element of D, $(A;-) \in D_-$ is improper element of D, the elements $(A;+) = (A;-)$ with $A \in S_0$ are called degenerate. The first component A of the element $(A;\alpha) \in D$ is called the *proper projection* of $(A;\alpha)$ further denoted $\mathrm{pro}(A;\alpha)$ and the second component α is the *direction* of $(A;\alpha)$. Symbolically, pro: $D \longrightarrow S$, with $\mathrm{pro}(A;\alpha) = A$, $A \in S$, $\alpha \in \Lambda$, and $\tau : D \longrightarrow \Lambda$, with $\tau(A;\alpha) = \alpha$, $A \in S_+$, $\alpha \in \Lambda$; $\tau(A;\alpha) = +$, $A \in S_0$. The components A, α can be viewed as "projections" of $(A;\alpha) \in D = S \otimes \Lambda$ on the "coordinate axes" S, Λ. In D we define the automorphism dual : $D \longrightarrow D$ by dual $(A;\alpha) = (A;-\alpha)$.

Proposition 2.1. *Every semigroup* $(S, S_0, +, -) \in \mathcal{I}$ *can be isomorphically embedded in a group* $(D, +)$*, where* $D = S \otimes \Lambda$ *is the set of directed elements generated by* S*, and* $(D, +) \cong (\mathcal{G}, +)$*.*

Sketch of the proof. We first define a bijection $\psi : \mathcal{G}_{+,0} \longrightarrow D_{+,0}$ by

$$\psi(A, 0) = (A; +) \in D_{+,0},\ A \in S. \tag{4}$$

We define "+" in $D_{+,0}$ by $(A; +) + (B; +) = (A + B; +)$, then ψ is an isomorphism between the semigroups $(D_{+,0}, +)$ and $(\mathcal{G}_{+,0}, +)$ and $(S, +) \cong (D_{+,0}, +) \cong (\mathcal{G}_{+,0}, +)$. We then show that the isomorphism ψ can be uniquely extended over D by

$$\psi(0, B) = (-B; -) \in D_-,\ B \in S_+, \tag{5}$$

generating thus a group operation "+" in D; see (7) below. Note that (5) is valid for $B \in S_0$ as well (and coincides with (4) for $A = -B \in S_0$). □

The isomorphic embedding $\mu : S \longrightarrow D$ of the semigroup S in the group D can be defined as the mapping $\mu = \varphi\psi$, where φ is the embedding $\varphi(A) = (A, 0)$ of S in $\mathcal{G}$, and ψ is the bijection between $\mathcal{G}$ and D defined by (4) and (5). For the algebraic completion of the semigroup S up to a directed group $(D, +)$ we substantially make use of the negation operator in S; see (5). The formula $\mathrm{opp}(B; \alpha) = (-B; -\alpha)$ shows that we cannot express the opposite elements in D without negation.

The embedding φ extends the operator negation from S to $\mathcal{G}$ by $\mathrm{neg}(A, B) = (-A, -B)$, $A, B \in S$, and the isomorphism ψ defines negation in D by $\mathrm{neg}(A; \alpha) = (-A; \alpha)$, $A \in S$. We defined in D the operator $\mathrm{dual}(A; \alpha) = (A; -\alpha)$; the operator dual in $\mathcal{G}$ generated by the isomorphism between $\mathcal{G}$ and D is $\mathrm{dual}(A, B) = (-B, -A)$.

We shall use boldface letters for a brief notation of the elements of the groups D and $\mathcal{G}$, generated by $S \in \mathcal{I}$. Using such notation we can write

$$\mathrm{neg}\ (\mathbf{A}) = \mathrm{opp}\ (\mathrm{dual}\ \mathbf{A}) = \mathrm{dual}\ (\mathrm{opp}\ \mathbf{A}),\ \mathbf{A} \in D\ (\mathbf{A} \in \mathcal{G}). \tag{6}$$

We see that besides opp we have two more automorphisms in the group $\mathcal{G}$, resp. D: dual and neg. The operation "+" in D generated by the isomorphism ψ is given by:

$$(A; \alpha) + (B; \beta) = (A +^{\alpha\beta} B; \tau((A; \alpha) + (B; \beta))),\ A, B \in S,\ \alpha, \beta \in \Lambda, \tag{7}$$

wherein $+^+ = +$ is the outer addition, $+^-$ is the inner addition (1), and $\tau((A; \alpha) + (B; \beta)) = \{\alpha,$ if $\alpha = \beta$ or $\alpha = -\beta,\ A >_\omega B;\ \beta,$ if $\alpha = -\beta,\ A <_\omega B;\ +,$ if $\alpha = -\beta,\ A =_\omega B\}$. We can obtain explicit expressions similar to (7) for various composite operations in D, involving addition, negation, opposite and dual, such as $(A; \alpha) + \mathrm{neg}(B; \beta)$, $(A; \alpha) + \mathrm{dual}(B; \beta)$, etc. We can thus reformulate any expression from $\mathcal{G}$ in terms of $D = S \otimes \Lambda$, that is in terms of S (and Λ). As an example let us "project" the associative law in $\mathcal{G}$, resp. D, it into S. Substituting $\mathbf{A} = (A; \alpha)$, $\mathbf{B} = (B; \beta)$, $\mathbf{C} = (C; \gamma)$ in $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$, using (7), we obtain $(A +^{\alpha\beta} B; \tau(\mathbf{A} + \mathbf{B})) + (C; \gamma) = (A; \alpha) + (B +^{\beta\gamma} C; \tau(\mathbf{B} + \mathbf{C}))$. Comparing the proper projections of both sides we obtain the following conditionally associative law in $(S, +, -) \in \mathcal{I}$:

Proposition 2.2. *For $A, B, C \in S \in \mathcal{I}$, $\alpha, \beta, \gamma \in \{+, -\}$, we have $(A +^{\alpha\beta} B) +^{\gamma\mu} C = A +^{\alpha\nu} (B +^{\beta\gamma} C)$, where $\mu = \tau((A;\alpha) + (B;\beta))$, $\nu = \tau((B;\beta) + (C;\gamma))$.*

3 Interval Semigroups and their Completion

The interval arithmetic systems (involving interval vectors, functions etc.) are special case of semigroups from $\mathcal{I}$ with resp. to addition/multiplication and possess unique negation/reciprocal operator. We consider briefly the simplest interval semigroup involving the set of compact intervals over the real line. For $a, b \in \mathbb{R}$ with $a \le b$ the set $[a, b] = \{x \mid x \in \mathbb{R}, a \le x \le b\}$ is called an interval. Denote the set of all intervals by $I(\mathbb{R})$. The sum of $[a, b]$, $[c, d] \in I(\mathbb{R})$ is the interval $\{u + v \mid u \in [a, b], v \in [c, d]\} = \{z \mid a + c \le x \le b + d\}$, that is $[a, b] + [c, d] = [a + c, b + d]$. The set $S = I(\mathbb{R})$ is a semigroup under the addition thus defined. The set of reals $S_0 = \mathbb{R}$ is isomorphic to the set of degenerate intervals $[a, a]$ and is thus a subgroup of $I(\mathbb{R})$. The width of $[a, b]$ is $w([a, b]) = b - a$. For $[a, b]$, $[c, d] \in I(\mathbb{R})$, the equation $[a, b] + [x, y] = [c, d]$ has a unique solution for $[x, y]$ iff $w([a, b]) \le w([c, d])$, or briefly, $[a, b] \le_w [c, d]$; if $[c, d] \le_w [a, b]$, then the equation $[c, d] + [u, v] = [a, b]$ is uniquely solvable. The relation $[a, b] =_w [c, d] \Longleftrightarrow w([a, b]) = w([c, d])$ generates equivalence classes in $I(\mathbb{R})$ which are cosets of the form $[a, b] + \mathbb{R} = \{[a, b] + t \mid t \in \mathbb{R}\}$. The operator negation, defined by $-[a, b] = [-b, -a]$, is the unique operator in $I(\mathbb{R})$ satisfying the conditions of Definition 1.2. The operation (outer) subtraction "$-$" in $I(\mathbb{R})$ is $[a, b] - [c, d] = [a, b] + (-[c, d]) = [a - d, b - c]$. From (1) we obtain that the operations inner addition "$+^-$" and inner subtraction "$-^-$" are expressed resp. by

$$[a,b] +^- [c,d] \;=\; \begin{cases} [a+d, b+c], & [a,b] \ge_w [c,d], \\ [b+c, a+d], & [a,b] \le_w [c,d]; \end{cases}$$

$$[a,b] -^- [c,d] \;=\; \begin{cases} [a-c, b-d], & [a,b] \ge_w [c,d], \\ [b-d, a-c], & [a,b] \le_w [c,d]. \end{cases}$$

The interval semigroup $(I(\mathbb{R}), +) \in \mathcal{I}$ can be embedded in the group $(\mathcal{G}(\mathbb{R}), +)$, with $\mathcal{G}(\mathbb{R}) = I(\mathbb{R})^2 \setminus E = \{([a, b], 0) \mid a \le b\} \bigcup (0, [a, b]) \mid a \le b\}$, which is isomorphic to the set of couples of real numbers $\mathbb{I}(\mathbb{R}) = \{[a, b] \mid a, b \in \mathbb{R}\}$, called generalized intervals [1], with $[a, b] + [c, d] = [a + c, b + d]$. The semigroup $(I(\mathbb{R}), +, -)$ can be also embedded in the group $(D(\mathbb{R}), +)$, where $D(\mathbb{R}) = \{[A; \alpha] \mid A \in I(\mathbb{R}), \alpha = \pm\}$ is the set of directed intervals with addition defined by (7). The basic automorphisms in $(\mathbb{I}(\mathbb{R}), +)$ are $\mathrm{dual}[a, b] = [b, a]$, $-[a, b] = [-b, -a]$ and $\mathrm{opp}[a, b] = [-a, -b]$, $a, b \in \mathbb{R}$. In $(D(\mathbb{R}), +)$ we have $\mathrm{dual}[A; \alpha] = [A; -\alpha]$, $-[A; \alpha] = [-A; \alpha]$ and $\mathrm{opp}[A; \alpha] = [-A; -\alpha]$, $A \in I(\mathbb{R})$, $\alpha = \pm$. In a dual manner we embed the multiplicative interval semigroup $(I(\mathbb{R})^*, \times, /) \in \mathcal{I}$, with $I(\mathbb{R})^* = \{[a, b] \mid a \le b < 0 \text{ or } 0 < a \le b\}$, $[a, b] \times [c, d] = \{uv \mid u \in [a, b], v \in [c, d]\}$, and reciprocal operator "/" given by $1/[a, b] = [1/b, 1/a]$, $[a, b] \in I(\mathbb{R})^*$, into the groups $(\mathbb{I}(\mathbb{R})^*, \times)$ [1], resp. $(D(\mathbb{R})^*, \times)$ [2]. Associative and distributive relations in $(\mathbb{I}(\mathbb{R}), +, \times)$, resp. $(D(\mathbb{R}), +, \times)$ can be derived and "projected" in $(I(\mathbb{R}), +, -, \times, /)$. For example, the distributive-like relation $(\mathbf{A}+\mathbf{B})\times\mathbf{C}_{\sigma(A+B)} = (\mathbf{A}\times\mathbf{C}_{\sigma(A)}) + (\mathbf{B}\times\mathbf{C}_{\sigma(B)})$ in $\mathbb{I}(\mathbb{R})^*$ can be translated for normal (proper) intervals via passing through directed intervals; see [2].

4 Conclusion

We study the semigroup $S \in \mathcal{I}$, which generates: a group $\mathcal{G} = S^2 \setminus E$ of ordered couples of elements of S and a group $D = S \otimes \Lambda$ of directed elements. We find an isomorphism between the two groups. Our results generalize results related to interval arithmetic [2], and their application to interval algebraic equations [3], [5], and computation of functional ranges [2], [4]. The operator negation/reciprocal together with the outer and inner addition/multiplication and subtraction/division involved, can be used for the solution of algebraic problems formulated in S (where no inverse elements exist), which corresponds to the solution of algebraic problems in the extended sets $\mathcal{G}$ and D using inverse elements. This relates algebraic problems formulated in S and corresponding algebraic problems in the extended sets, which can be solved by means of group operations. To interpret the solutions in the extended spaces we can relate them to the solutions of the original problems, formulated in S, using the transition technique described in this work. Without such technique the extended interval space $\mathbb{I}(\mathbb{R})$, although known for two decades [1], remains useless for applications.

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