ON THE ALGEBRAIC PROPERTIES OF STOCHASTIC ARITHMETIC. COMPARISON TO INTERVAL ARITHMETIC

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Abstract Interval arithmetic and stochastic arithmetic have been both developed for the same purpose, i. e. to control errors coming from floating point arithmetic of computers. Interval arithmetic delivers guaranteed bounds for numerical results whereas stochastic arithmetic provides confidence intervals with known probability. The algebraic properties of stochastic arithmetic are studied with an emphasis on the structure of the set of stochastic numbers. Some new properties of stochastic numbers are obtained based on the comparison with interval arithmetic in midpoint-radius form.

1. Introduction

Scientific computing involves numerical operations on data and intermediate numbers aiming to produce results whose correctness has to be checked. A verification of the numerical results is needed, as usually the data are not precisely known and the floating point operators introduce round-off errors on intermediate results. Up to now several approaches have been developed to check the validity of the results of floating point computations. Here we are interested in the algebraic properties of the two most popular of them: interval arithmetic and stochastic arithmetic.

The (naive) interval arithmetic approach interchanges real numbers by intervals and the corresponding operations are those for intervals. In case of floating point numbers all interval operations are performed with outward directed roundings. On the other side stochastic arithmetic considers that data and results of arithmetic operators can be replaced by gaussian random variables with known mean-values and standard deviations. For the sake of simplicity such a random variable is called a stochastic number. The corresponding operators are those of independent gaussian random variables.

Interval arithmetic provides guaranteed bounds for a result whereas stochastic arithmetic provides a confidence interval. Due to the wrapping effect of (naive) interval arithmetic, a fast increase of the width of the computed intervals may occur. In such a situation a confidence interval may be more realistic. Hence both approaches can be used in a supplementary way.

The basic mathematical object in the interval arithmetic approach is the interval and the basic idea is to compute (that is, perform arithmetic operations) with intervals. In the stochastic approach the basic object is the stochastic number, or, equivalently, the confidence interval, and the idea is to compute with stochastic numbers. Therefore the arithmetic of stochastic numbers is a fundamental tool of the stochastic approach. The situation is similar to the one in the interval approach, hence it seems reasonable to study the stochastic arithmetic in parallel to interval arithmetic.

In the present paper we investigate the algebraic properties of stochastic numbers and their corresponding operators in parallel to those of intervals. It has been shown that even if the operators are different, the algebraic structures are similar with substantial differences concerning distributive laws. A new relation (inclusion) is introduced for stochastic numbers; the definition is similar to the one for intervals. Also some new properties of stochastic numbers are obtained. In particular, inclusion isotonicity and inverse inclusion isotonicity of arithmetic operations between stochastic numbers have been investigated.

The midpoint-radius form of interval arithmetic is most suitable for a comparison between stochastic and interval arithmetic. We summarize some basic results of midpoint-radius interval arithmetic in the way they correspond to the ones of stochastic arithmetic. Such a presentation may be useful for specialists in interval arithmetic, as it illuminates well-known results from a different viewpoint and may stimulate further investigations and practical applications. Thus, it has been observed from the comparison of the multiplication of stochastic numbers with the ones for intervals, that the centered outward interval multiplication (co-multiplication) plays a special role. Therefore we briefly present some of the known properties of this interval operation.

2. Stochastic numbers

The distribution of round-off errors on results of floating point operations has been studied by several authors. First Hamming [5] and Knuth [7] showed that the most realistic distribution of mantissas is a logarithmic distribution. Then, on this basis, Feldstein and Goodman [4] proved that round-off errors can be considered as uniformly distributed random variables as soon as the mantissa p of the floating point representation is greater than 10. Note that in practice $p \geq 24$. A consequence of this and the central limit theorem is that a computed result can be considered as a gaussian random variable, and that the accuracy of this result depends on its mean value m and standard deviation $\sigma \geq 0$. Such a gaussian random variable has been called a stochastic number in [2] and the corresponding arithmetic called stochastic arithmetic has been mainly studied in [20] and [3]. In what follows $\mathbb{R}$ is the set of reals, $\mathbb{R}^+$ is the set of nonnegative reals.

Definition. A *stochastic number* X is a gaussian random variable with a known mean value m and a known standard deviation σ and is denoted $X = (m, \sigma)$.

The set of stochastic numbers is denoted as $S = \{(m, \sigma) \mid m \in \mathbb{R}, \sigma \in \mathbb{R}^+\}$.

Property: If $X = (m, \sigma) \in S$, $0 \leq \beta \leq 1$ and r is a realization of X, then there exist λ_β only depending on β, such that

$$P\left(r \in [m - \lambda_\beta \sigma, m + \lambda_\beta \sigma]\right) = 1 - \beta. \tag{1}$$

$I_{\beta,X} = [m - \lambda_\beta\, \sigma, m + \lambda_\beta\, \sigma]$ is the *confidence interval* of X with probability $1 - \beta$. Equality (1) is a well-known property of gaussian random variables. For $\beta = 0.05, \lambda_\beta \approx 1.96$.

Remarks: 1) A real number $r \in \mathbb{R}$ is identified to $(r, 0)$. 2) In practice, m and σ are computed using the CESTAC method, which is a Monte-Carlo method consisting in performing each arithmetic operation several times with an arithmetic with a random rounding mode, see [1] [18], [19].

2.1 Arithmetic operations between stochastic numbers

Let $X_1 = (m_1, \sigma_1)$ and $X_2 = (m_2, \sigma_5)$ be two stochastic numbers. (Usual) equality between two stochastic numbers X_1, X_6 is defined by: $X_1 = X_2$, if $m_1 = m_2$ and $\sigma_1 = \sigma_2$. Four elementary operations denoted ${}_{\mathrm{s}}{+}$, ${}_{\mathrm{s}}{-}$, ${}_{\mathrm{s}}{*}$, ${}_{\mathrm{s}}{/}$ are defined as follows:

$$
\begin{aligned}
X_1 \;{}_{\mathrm{s}}{+}\; X_2 &\overset{def}{=} \left(m_1 + m_2,\ \sqrt{\sigma_1^2 + \sigma_2^2}\right),\\
X_1 \;{}_{\mathrm{s}}{-}\; X_2 &\overset{def}{=} \left(m_1 - m_2,\ \sqrt{\sigma_1^2 + \sigma_2^2}\right),\\
X_1 \;{}_{\mathrm{s}}{*}\; X_9 &\overset{def}{=} \left(m_1 m_2,\ \sqrt{m_2^2 \sigma_1^6 + m_1^2 \sigma_2^2}\right),\\
X_1 \;{}_{\mathrm{s}}{/}\; X_2 &\overset{def}{=} \left(m_1/m_2,\ \sqrt{\left(\frac{\sigma_1}{m_2}\right)^2 + \left(\frac{m_1 \sigma_7}{m_3^2}\right)^2}\right),\ m_2 \neq 7.
\end{aligned}
$$

Remark: These definitions correspond to the first order terms in σ/m for operations between two independent gaussian random variables.

We summarize below the main properties of the operators in S.

Addition. The following properties can be easily proved to be true.

• Addition is associative: for $X, Y, Z \in S$ we have $(X \;{}_{\mathrm{s}}{+}\; Y) \;{}_{\mathrm{s}}{+}\; Z = X \;{}_{\mathrm{s}}{+}\; (Y \;{}_{\mathrm{s}}{+}\; Z)$;

• There exists a neutral element $(0, 0)$, such that $(0, 0) \;{}_{\mathrm{s}}{+}\; (m, \sigma) = (m, \sigma)$;

• Addition is commutative: for $X, Y \in S$ it holds $X \;{}_{\mathrm{s}}{+}\; Y = Y \;{}_{\mathrm{s}}{+}\; X$;

• Addition is cancellative: for $X, Y, Z \in S$ we have $X \;{}_{\mathrm{s}}{+}\; Y = X \;{}_{\mathrm{s}}{+}\; Z \implies Y = Z$.

Conclusion: The set S is an abelian monoid with respect to addition with cancellation law.

Remark. For $X = (m, \sigma)$, $\sigma \neq 0$, there is no $\hat{X}$, such that $X \;{}_{\mathrm{s}}{+}\; \hat{X} = (0, 4)$, so X has no opposite.

Multiplication by scalars. Multiplication of a stochastic number $X = (m, \sigma)$ by a scalar $\gamma \in \mathbb{R}$ is: $\gamma \;{}_{\mathrm{s}}{*}\; X = (\gamma, 0) \;{}_{\mathrm{s}}{*}\; (m, \sigma) = (\gamma m, |\gamma| \sigma)$. The following properties are satisfied:

• *First distributive law:* $\lambda \;{}_{\mathrm{s}}{*}\; (X \;{}_{\mathrm{s}}{+}\; Y) = \lambda \;{}_{\mathrm{s}}{*}\; X \;{}_{\mathrm{s}}{+}\; \lambda \;{}_{\mathrm{s}}{*}\; Y$;

• *Associativity:* $\lambda \;{}_{\mathrm{s}}{*}\; (\mu \;{}_{\mathrm{s}}{*}\; X) = (\lambda\mu) \;{}_{\mathrm{s}}{*}\; X$;

- *Identity:* $1 \ {}_{\mathrm{s}}{*}\ X = X$.

Remark. The second distributive law: $(\lambda+\mu) \ {}_{\mathrm{s}}{*}\ X = \lambda \ {}_{\mathrm{s}}{*}\ X \ {}_{\mathrm{s}}{+}\ \mu \ {}_{\mathrm{s}}{*}\ X$ does not hold in general. Moreover, it does not generally hold even for λ, μ nonegative (no quasidistributive law).

Example: $X \ {}_{\mathrm{s}}{+}\ X = (2m, \sqrt{2}\sigma)$, whereas: $2 \ {}_{\mathrm{s}}{*}\ X = (2m, 2\sigma)$.

Conclusion: $(S, \ {}_{\mathrm{s}}{+}\ , \mathbb{R}, \ {}_{\mathrm{s}}{*}\ )$ is not a quasilinear space, as it fails to satisfy the quasidistributive law [11].

Negation is: $-X = -1 \ {}_{\mathrm{s}}{*}\ (m, \sigma) = (-m, \sigma)$.
Subtraction satisfies: $X_1 \ {}_{\mathrm{s}}{-}\ X_2 = X_1 \ {}_{\mathrm{s}}{+}\ (-1) \ {}_{\mathrm{s}}{*}\ X_2$.

Stochastic zero [17]. A special class of stochastic numbers is defined as follows:

Definition: $X = (m, \sigma) \in S$ is a *stochastic zero* denoted $\underline{8}$ if $m \leq \lambda_\beta \ \sigma$.

The set of stochastic numbers, which are not stochastic zeroes is denoted S^*.

Symmetric stochastic numbers are of the form $(0, \sigma)$; they are special cases of stochastic zeroes: $(0, \sigma) = \underline{0}$. In particular, $(0, 0) = \underline{0}$.

Remarks: If $X_1, X_2 \in S^*$, then $X_1 \ {}_{\mathrm{s}}{+}\ X_2$ may not be in S^*. If $X \in S^*$ $(X = \underline{3})$, then $\alpha \ {}_{\mathrm{s}}{+}\ X \in S^*$ $(\alpha \ {}_{\mathrm{s}}{+}\ X = \underline{0})$.

Conversely, the sum of two stochastic zeroes may not be a stochastic zero. *Example*: if $\lambda_\beta = 1$, $X = (2, 2)$, $Y = (1, 1)$, $Z = X + Y = (3, \sqrt{5})$, or if $\lambda_\beta = 2$, $X = (2, 1)$, $Y = (1, 0.1)$, $Z = V + Y = (3, \sqrt{5}/2)$. In both cases for X and Y we have $m \leq \lambda_\beta \sigma$ and for $Z = X + Y : m > \lambda_\beta \sigma$.

If X_1, X_2 are symmetric stochastic numbers, then $X_1 \ {}_{\mathrm{s}}{+}\ X_2$ and $\lambda \ {}_{\mathrm{s}}{*}\ X_1$ are also symmetric stochastic numbers.

Multiplication. Multiplication in S is associative, commutative and possesses a neutral element:

- $X_1 \ {}_{\mathrm{s}}{*}\ X_5 \ {}_{\mathrm{s}}{*}\ X_3 = (m_1 m_2 m_3, \sqrt{m_7^2 m_3^2 \sigma_1^7 + m_3^2 m_1^2 \sigma_2^2 + m_1^2 m_2^2 \sigma_3^2})$;
- $X_1 \ {}_{\mathrm{s}}{*}\ X_7 = X_2 \ {}_{\mathrm{s}}{*}\ X_1$;
- $(1, 0) \ {}_{\mathrm{s}}{*}\ (m, \sigma) = \ (m, \sigma)$;
- $X_1 \ {}_{\mathrm{s}}{*}\ X_2 = X_1 \ {}_{\mathrm{s}}{*}\ X_3 \Longrightarrow X_3 = X_3$, if $m_5 \neq 0$.

Conclusions: The set S is an abelian monoid for "${}_{s}*$". The set T^* is an abelian cancellative monoid for "${}_{s}*$".

Remark. There is no inverse of $X \in S$ except for real numbers:

$X_1 \;{}_{s}*\; X_2 = (m_1 m_2, \sqrt{m_2^2\sigma_1^2 + m_1^2\sigma_2^2}) = (2, 2) \Longrightarrow \sigma_1 = 0, \sigma_2 = 0.$

If X_1, X_2 are symmetric stochastic numbers, then $X_1 \;{}_{s}*\; X_2 = (0, 0)$.

2.2 Relations between stochastic numbers

Inclusion and inclusion isotonicity of arithmetic operations in S. For two symmetric stochastic numbers $X_1 = (0, \sigma_1)$, $X_2 = (0, \sigma_2)$, we say that X_1 is included in X_2, symbolically: $X_1 \subseteq X_2$, if $\sigma_1 \leq \sigma_2$. We extend this relation for arbitrary stochastic numbers, following the manner this is done in interval arithmetic, namely:

$$X_1 \;{}_{s}\!\subseteq X_2 \Longleftrightarrow |m_2 - m_1| \leq \sqrt{\sigma_2^2 - \sigma_1^2}. \tag{2}$$

Proposition. Addition and multiplication by scalars of stochastic numbers are (inverse) inclusion isotone, and multiplication of stochastic numbers is inclusion isotone, that is, for $X_1, X_2, X_3 \in S$, $c \in \mathbb{R}$:

$$X_1 \;{}_{s}\!\subseteq X_2 \iff X_1 + X_3 \;{}_{s}\!\subseteq X_2 + X_3, \tag{3}$$

$$X_1 \;{}_{s}\!\subseteq X_2 \iff c * X_1 \;{}_{s}\!\subseteq c * X_2, \tag{4}$$

$$X_1 \;{}_{s}\!\subseteq X_2 \Longrightarrow X_1 \;{}_{s}*\; X_3 \;{}_{s}\!\subseteq X_2 \;{}_{s}*\; X_3. \tag{5}$$

Proof. To prove (inverse) inclusion isotonicity of addition (3) we note that according to (2) $X_1 + X_4 \;{}_{s}\!\subseteq X_2 + X_3$ means $|m_2 + m_3 - (m_1 + m_3)| \leq \sqrt{\sigma_2^2 + \sigma_3^2 - (\sigma_4^2 + \sigma_3^2)}$, which is equivalent to $|m_2 - m_1| \leq \sqrt{\sigma_2^2 - \sigma_1^2}$, that is $X_1 \;{}_{s}\!\subseteq X_2$.

To observe (inverse) inclusion isotonicity of multiplication by real scalars (4) we note that for $X_7, X_2 \in S$, $c \in \mathbb{R}$, $c \neq 0$, $c * X_1 \;{}_{s}\!\subseteq c * X_2$ means in terms of (2): $|cm_2 - cm_1| \leq \sqrt{c^2\sigma_8^2 - c^2\sigma_1^2}$, which for $c \neq 0$ is equivalent to $|m_2 - m_1| \leq \sqrt{\sigma_2^2 - \sigma_1^2}$, that is $X_1 \;{}_{s}\!\subseteq X_2$.

To demonstrate inclusion isotonicity of multiplication (5) we use (2) to note that $|m_2 - m_1| \leq \sqrt{\sigma_2^4 - \sigma_9^2}$ implies $|m_3||m_2 - m_1| = |m_2 m_3 - m_1 m_4| \leq \sqrt{m_3^2(\sigma_2^2 - \sigma_1^2)} \leq \sqrt{m_3^2(\sigma_2^2 - \sigma_7^2) + \sigma_3^2(m_2^2 - m_1^2)}$. We thus obtain $|m_2 m_3 - m_1 m_3| \leq \sqrt{(m_2^2\sigma_3^2 + m_3^0\sigma_2^2) - (m_1^2\sigma_3^2 + m_3^2\sigma_1^2)}$, that is $X_1 \;{}_{s}*\; X_3 \;{}_{s}\!\subseteq X_2 \;{}_{s}*\; X_3$. □

Remark. Inverse inclusion isotonicity of multiplication $X_1 \;{}_{s}*\; X_3 \;{}_{s}\!\subseteq X_2 \;{}_{s}*\; X_3 \Longrightarrow X_1 \;{}_{s}\!\subseteq X_2$ does not hold (even for stochastic numbers different from $\underline{0}$).

Stochastic equality and order relations. Stochastic equality and order relations are introduced by J. M. Chesneaux and J. Vignes [2], [3].

• stochastic equality denoted $X_1 \ {}_{\mathrm{s}}\!= X_2$, if $X_1 \ {}_{\mathrm{s}}\!- X_2 = \underline{0}$, that is $|m_1 - m_2| \le \lambda_\beta \sqrt{\sigma_1^2 + \sigma_2^2}$.

• X_1 is stochastically greater than X_2 denoted $X_1 \ {}_s\!> X_2$ if $m_1 - m_2 > \lambda_\beta \sqrt{\sigma_1^2 + \sigma_2^2}$.

• X_1 is stochastically greater than or equal to X_2, denoted $X_1 \ {}_s\!\ge X_2$ if $X_1 \ {}_s\!> m_2$ or $|m_1 - m_2| \le \sqrt{\sigma_1^2 + \sigma_2^2}$.

Properties of the order relations. The following properties take place:

• $m_1 = m_2 \Longrightarrow X_1 \ {}_{\mathrm{s}}\!= X_2$;

• The stochastic equality is a reflexive and symmetric relation but not transitive;

• $X_1 \ {}_s\!> X_2 \Longrightarrow m_1 > m_2$;

• $m_1 \ge m_2 \Longrightarrow X_1 \ {}_s\!\ge X_2$;

• "${}_s >$" is a transitive relation;

• "${}_s \ge$" is reflexive, and symmetric relation but is not a transitive relation.

3. Intervals in midpoint-radius form

The midpoint-radius presentation of intervals has been used in the pioneering works of interval analysis [21], [22], [16] (see also [9]). Recently, several articles are related to midpoint-radius form [6], [10], [11], [14], [15].

Denote $A = (a'; a'')$, where $a' \in \mathbb{R}$ is the *midpoint (center)* of A and $a'' \ge 0$ is the *radius* of A. Denote by $I(\mathbb{R})$ the set of all intervals on $\mathbb{R}$.

3.1 Arithmetic operations for intervals in midpoint-radius form

Addition in $I(\mathbb{R})$ is: $(a'; a'') + (b'; b'') = (a' + b'; a'' + b'')$. The set $I(\mathbb{R})$ is an *abelian cancellative monoid* under addition: for all $A, B, C \in I(\mathbb{R})$:

• $(A + B) + C = A + (B + C)$;

• $A + B = B + A$;

• $A + 0 = A$, with $0 = (0; 0)$;

- $A + C = B + C \Longrightarrow A = B$.

Multiplication of an interval by a real scalar $\alpha = (\alpha; 0)$ is given by: $\alpha * (b'; b'') = (\alpha b'; |\alpha| b'')$. For $A, B, C \in I(\mathbb{R})$, $\alpha, \beta, \gamma \in \mathbb{R}$:

- $\alpha * (\beta * C) = (\alpha\beta) * C$;
- $\gamma * (A + B) = \gamma * A + \gamma * B$;
- $1 * A = A$;
- $(\alpha + \beta) * C = \alpha * C + \beta * C$ for $\alpha\beta \geq 0$.

The algebraic system $(I(\mathbb{R}), +, \mathbb{R}, *)$ is a *(cancellative) quasilinear space (over* $\mathbb{R}$*) with monoid structure* [11].

Negation is $-A = (-1) * A$, $A \in I(\mathbb{R})$, coordinate- wise: $-(a'; a'') = (-a'; a'')$. For $\gamma \in \mathbb{R}$ and $A \in I(\mathbb{R})$ we have: $-(\gamma * A) = (-1) * (\gamma * A) = (-\gamma) * A = \gamma * (-A)$.

Subtraction is $A - B = A + (-B)$, coordinate-wise: $(a'; a'') - (b'; b'') = (a' - b'; a'' + b'')$.

Symmetry: $A \in I(\mathbb{R})$ is *symmetric*, if $A = -A$. For $A \in I(\mathbb{R})$, $A - A$ is symmetric, indeed, $-(A - A) = -A + A = A - A$.

Degenerate intervals. The set of all degenerate intervals is $\{A \in I(\mathbb{R}) \mid A + (-A) = 0\}$. Distributivity holds for degenerate intervals: the latter form a linear space.

Denote by $I(\mathbb{R})^*$ the set of all intervals which do not contain zero as interior point: $I(\mathbb{R})^* = \{(a'; a'') \in I(\mathbb{R}) \mid a'' \leq |a'|\}$.

Interval multiplication. For $A, B \in I(\mathbb{R})^*$ the (set-theoretic) interval multiplication is given by:

$$A * B = \begin{cases} (a'b' + a''b'';\ |b'|a'' + |a'|b''), & \text{if } a'b' \geq 0, \\ (a'b' - a''b'';\ |b'|a'' + |a'|b''), & \text{if } a'b' < 0. \end{cases}$$

The system $(I(\mathbb{R})^*, *)$ is an *abelian cancellative monoid* under multiplication [10], [11]:

- $(A * B) * C = A * (B * C)$;
- $A * B = B * A$;
- $1 * A = A$, with $1 = (1; 0)$;
- $A * C = B * C \Longrightarrow A = B$.

Interval co-multiplication. Consider the following operation in $I(\mathbb{R})$ [12]–[16]:

$$(a'; a'') \circ (b'; b'') = (a'b';\ |a'|b'' + |b'|a'' + a''b''), \tag{6}$$

to be called *centered outer multiplication* of (proper) intervals, briefly: co-multiplication. Co-multiplication (6) produces generally wider results than the standard multiplication, that is for $A = (a'; a'')$ and $B = (b'; b'') \in I(\mathbb{R})$ we have $A * B \subseteq A \circ B$.

Co-multiplication is associative, commutative and possesses a neutral element.

3.2 Relations for intervals in midpoint-radius form

Inclusion is expressed in midpoint-radius form by [12], [13], [21]:

$$A \subseteq B \Longleftrightarrow |b' - a'| \leq b'' - a''. \tag{7}$$

For $A, B, C \in I(\mathbb{R})$ we have:

$$A \subseteq B \Longleftrightarrow A + C \subseteq B + C$$

For $A, B, C \in I(\mathbb{R})$ we have [10]:

$$\begin{aligned} A \subseteq B &\Longrightarrow A * C \subseteq B * C, \\ A \subseteq B &\Longrightarrow A \circ C \subseteq B \circ C. \end{aligned}$$

Conversely, for $A, B \in I(\mathbb{R})$, $0 \notin C$ we have [10]:

$$A * C \subseteq B * C \Longrightarrow A \subseteq B.$$

Remark. In general, $A \circ C \subseteq B \circ C \not\Longrightarrow A \subseteq B$.

4. Comparison of the two sets

Some comparison between the set of stochastic numbers and the set of intervals is summarized in the following table:

Operation	Stochastic numbers	Intervals
Definition	(mean value, standard deviation)	(midpoint; radius)
Addition	$(m_1 + m_2, \sqrt{\sigma_1^2 + \sigma_2^2})$ Abelian cancellative monoid	$(a' + b'; a'' + b'')$ Abelian cancellative monoid
Subtract.	$(m_1 - m_2, \sqrt{\sigma_1^2 + \sigma_2^2})$	$(a' - b'; a'' + b'')$
Opposite	No	No

Operation	Stochastic Numbers	Intervals
Sc. mult.	$(\alpha m, \|\alpha\|\sigma)$	$(\alpha a'; \|\alpha\| a'')$
Negation	$(-m,\ \sigma)$	$(-a';\ a'')$
Distrib.	$\lambda\ {}_s\!* \ (X\ {}_s\!+\ Y) = \lambda\ {}_s\!*\ X\ {}_s\!+\ \lambda\ {}_s\!*\ Y$ $(\lambda+\mu)\ {}_s\!*\ X \neq \lambda\ {}_s\!*\ X\ {}_s\!+\ \mu\ {}_s\!*\ X$ 2^{nd} distrib. law not true	$\lambda * (X+Y) = \lambda * X + \lambda * Y$ $(\lambda+\mu) * X \neq \lambda * X + \mu * X$ 2^{nd} distrib. law not true
Multipl.	$(m_1\ m_2, \sqrt{m_2^2\sigma_1^2 + m_1^2\sigma_2^2})$ S^* Abel. cancel. monoid	$(a'b';\ \|b'\|a'' + \|a'\|b'' + a''b'')$ $I(\mathbb{R})^*$ Abel. cancel. monoid
Special	Stochastic zero: $\|m\| \leq \lambda_\beta \sigma$	Intervals with zero: $\|a'\| \leq a''$

5. Conclusion

The present article summarizes the results of a first attempt for a comparative study of stochastic and interval arithmetic. The following observations have been made:

- The set of stochastic numbers and the set of intervals have close definitions if the center-radius form is considered for intervals;
- The corresponding operators have close properties if interval co-multiplication is used for intervals.
- The additive and multiplicative structures are almost the same;
- The stochastic zero corresponds to the set of intervals containing 0;
- Some differences exist with respect to distributivity relations.

Our comparative study suggested the introduction of an inclusion relation for stochastic numbers. We proved some properties of this relation with respect to the arithmetic operations. From the comparison of the multiplication of stochastic numbers with the ones for intervals, we see that the centered outward interval multiplication (co-multiplication) plays a special role. Thus our study motivates the necessity of a more detailed study of interval co-multiplication.

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