On the numerical solution to linear problems using stochastic arithmetic

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ABSTRACT

It has been recently shown that computation with stochastic numbers as regard to addition and multiplication by scalars can be reduced to computation in familiar vector spaces. This result allows us to solve certain practical problems with stochastic numbers and to compare algebraically obtained results with practical applications of stochastic numbers, such as the ones provided by the CESTAC method. Such comparisons give additional information related to the stochastic behavior of random roundings in the course of numerical computations. A number of original numerical experiments are presented that agree with the expected theoretical results.

Categories and Subject Descriptors

F.1.2 [**Computation by Abstract Devices**]: Probabilistic Computation; G.1.3 [**Numerical Analysis**]: Numerical Linear Algebra—*error analysis, linear systems*; G.3 [**Probability and Statistics**]: Stochastic Processes; I.6.8 [**Simulation and Modeling**]: Monte Carlo

General Terms

Reliability, Experimentation, Theory, Verification

Keywords

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1. INTRODUCTION

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Stochastic numbers are gaussian random variables with a known mean value and a known standard deviation. Some fundamental properties of stochastic numbers are considered in [3], [8].

The mean values of the stochastic numbers satisfy the usual real arithmetic, whereas standard deviations are added and multiplied by scalars in a specific way. As regard to addition the system of standard deviations is an abelian monoid with cancellation law. This monoid can be embedded in an additive group and after a suitable extension of multiplication by scalars one obtains a so-called s-space, which is closely related to a vector space [1], [4]. This allows us to introduce in s-spaces concepts like linear combination, basis, dimension etc. Thus, in theory, computations in s-spaces are reduced to computations in vector spaces. This enables to find explicit expressions for the solution of certain algebraic problems involving stochastic numbers.

In practice, stochastic numbers are computed using the CESTAC method, which is a Monte-Carlo method consisting in performing each arithmetic operation several times using an arithmetic with a random rounding mode, see [2], [6], [7].

In Sections 2 we briefly present the main results of our theory of s-spaces as regard to the arithmetic operations for addition and multiplication by scalars needed for the purposes of this study; for a detailed presentation of the theory, see [4]. Section 3 considers the algebraic solution of linear systems of equations which right-hand sides involve stochastic numbers. In Section 4 we extend our idea from [5] to compare the algebraic solution of a problem involving stochastic numbers with the solution obtained numerically by the CESTAC method. Several numerical experiments are presented.

2. STOCHASTIC ARITHMETIC

By $\mathbb{R}$ we denote the set of reals; the same notation is used for the linearly ordered field of reals $\mathbb{R} = (\mathbb{R}, +, \cdot, \leq)$. For any integer $n \geq 1$ we denote by $\mathbb{R}^n$ the set of all n-tuples $(\alpha_1, \alpha_2, ..., \alpha_n)$, where $\alpha_i \in \mathbb{R}$. The set $\mathbb{R}^n$ forms a vector space under the familiar operations of addition and multiplication by scalars denoted by $V^n = (\mathbb{R}^n, +, \mathbb{R}, \cdot)$, $n \geq 1$. By $\mathbb{R}^+$ we denote the set of nonnegative real numbers.

A *stochastic number* $X = (m; s)$ is a gaussian random variable with mean value $m \in \mathbb{R}$ and (nonnegative) standard deviation $s \in \mathbb{R}^+$. The set of all stochastic numbers is $\mathbb{S} = \{(m; s) \mid m \in \mathbb{R}, s \in \mathbb{R}^+\}$.

The arithmetic for stochastic numbers. Let $X_1 = (m_1; s_1)$, $X_2 = (m_2; s_2) \in \mathbb{S}$. Addition and multiplication by scalars are defined by:

$$X_1 + X_2 = (m_1; s_1) + (m_2; s_2) \overset{def}{=} m_1 + m_2; \sqrt{s_1^2 + s_2^2},$$

$$\gamma * X = \gamma * (m; s) \overset{def}{=} \gamma m; |\gamma| s, \ \gamma \in \mathbb{R}.$$

A stochastic number of the form $(0; s)$, $s \in \mathbb{R}^+$, is called *symmetric*. If X_1, X_2 are symmetric stochastic numbers, then $X_1 + X_2$ and $\lambda * X_1$, $\lambda \in \mathbb{R}$, are also symmetric stochastic numbers. Thus there is a 1–1 correspondence between the set of symmetric stochastic numbers and the set $\mathbb{R}^+$. We shall use special symbols "$\oplus$", "$*$" for the arithmetic operations over standard deviations, as these operations are different from the corresponding ones for numbers. The operations "$\oplus$", "$*$" induce a special arithmetic on the set $\mathbb{R}^+$. Consider the system $(\mathbb{R}^+, \oplus, \mathbb{R}, *)$, where:

$$\alpha \oplus \beta = \sqrt{\alpha^2 + \beta^2}, \ \alpha, \beta \in \mathbb{R}^+, \quad (1)$$
$$\gamma * \delta = |\gamma| \delta, \ \gamma \in \mathbb{R}, \ \delta \in \mathbb{R}^+. \quad (2)$$

PROPOSITION 1. *[4] The system* $(\mathbb{R}^+, \oplus, \mathbb{R}, *)$ *is an abelian additive monoid with cancellation, such that for* $s, t \in \mathbb{R}^+$, $\alpha, \beta \in \mathbb{R}$:

$$\alpha * (s \oplus t) = \alpha * s \oplus \alpha * t, \quad (3)$$
$$\alpha * (\beta * s) = (\alpha\beta) * s, \quad (4)$$
$$1 * s = s, \quad (5)$$
$$(-1) * s = s, \quad (6)$$
$$\sqrt{\alpha^2 + \beta^2} * s = \alpha * s \oplus \beta * s, \ \alpha, \beta \geq 0. \quad (7)$$

A system satisfying the conditions of Proposition 1 is called an s-space of monoid structure.

2.1 The Group System of Standard Deviations

For $\alpha \in \mathbb{R}$ denote $\sigma(\alpha) = \{+, \text{ if } \alpha \geq 0; -, \text{ if } \alpha < 0\}$. We extend the operation addition "$\oplus$" for all $\alpha, \beta \in \mathbb{R}$, admitting thus negative reals, corresponding to *improper* standard deviations:

$$\alpha \oplus \beta \overset{def}{=} \sigma(\alpha + \beta)\sqrt{|\sigma(\alpha)\alpha^2 + \sigma(\beta)\beta^2|}. \quad (8)$$

Note that $\sigma(\alpha + \beta) = \sigma(\sigma(\alpha)\alpha^2 + \sigma(\beta)\beta^2) = \sigma(\alpha \oplus \beta)$ for $\alpha, \beta \in \mathbb{R}$.

From (8) the monoid $(\mathbb{R}^+, \oplus)$ into is thus isomorphically embedded in the system $(\mathbb{R}, \oplus)$, which is an abelian group with null 0 and opposite element $\text{opp}(\alpha) = -\alpha$, i. e. $\alpha \oplus (-\alpha) = 0$.

Indeed, from (8) we have
$$\alpha \oplus (-\alpha) = \sigma(\alpha - \alpha)\sqrt{|\sigma(\alpha)\alpha^2 - \sigma(\alpha)\alpha^2|} = \sigma(0)\sqrt{0} = 0.$$

Here are some examples of addition in the system $(\mathbb{R}, \oplus)$: $1 \oplus 1 = \sqrt{2}$, $1 \oplus 2 = \sqrt{5}$, $3 \oplus 4 = 5$, $4 \oplus (-3) = \sqrt{7}$, $3 \oplus (-4) = -\sqrt{7}$, $5 \oplus (-4) = 3$, $4 \oplus (-5) = -3$, $(-3) \oplus (-4) = -5$, $1 \oplus 2 \oplus 3 = \sqrt{14}$, $1 \oplus 2 \oplus (-3) = -2$.

Using (8) we obtain for $n \geq 2$

$$\alpha_1 \oplus ... \oplus \alpha_n = \sigma(\alpha_1 \oplus ... \oplus \alpha_n)\sqrt{|\sigma(\alpha_1)\alpha_1^2 + ... + \sigma(\alpha_n)\alpha_n^2|}. \quad (9)$$

PROPOSITION 2. *The equation* $\alpha_1 \oplus \alpha_2 \oplus ... \oplus \alpha_n = \beta$ *is equivalent to* $\sigma(\alpha_1)\alpha_1^2 + ... + \sigma(\alpha_n)\alpha_n^2 = \sigma(\beta)\beta^2$.

Multiplication by scalars is naturally extended on the set $\mathbb{R}$ of generalized standard deviations by: $\gamma * s = |\gamma|s$, $s \in \mathbb{R}$. Multiplication by -1 (negation) is $(-1) * s = |-1|s = s$, $s \in \mathbb{R}$. To avoid confusion we shall write the scalars always to the left side of the standard deviation. Under this convention we have, e. g. $(-2) * 2 = 4$, whereas $2 * (-2) = -4$. Note that if s is a standard deviation, then we have $\gamma * s = (-\gamma) * s$ for any $\gamma \in \mathbb{R}$.

It is easy to check that relations (3)–(7) hold true for generalized standard deviations. This justifies the following definition:

DEFINITION 1. *A system* $(\mathcal{S}, \oplus, \mathbb{R}, *)$, *such that:*
i) $(\mathcal{S}, \oplus)$ *is an abelian additive group, and*
ii) for any $s, t \in \mathcal{S}$ *and* $\alpha, \beta \in \mathbb{R}$ *relations (3)–(7) hold,*
is called an s-space over $\mathbb{R}$ *(of group structure).*

The canonical s-space. For any integer $k \geq 1$ the set $\mathcal{S} = \mathbb{R}^k$ of all k-tuples $(\alpha_1, \alpha_2, ..., \alpha_k)$ forms an s-space over $\mathbb{R}$ under the following operations

$$(\alpha_1, ..., \alpha_k) \oplus (\beta_1, ..., \beta_k) = (\alpha_1 \oplus \beta_1, ..., \alpha_k \oplus \beta_k), \quad (10)$$
$$\gamma * (\alpha_1, \alpha_2, ..., \alpha_k) = (|\gamma|\alpha_1, ..., |\gamma|\alpha_k), \quad (11)$$

where $\alpha_i \oplus \beta_i$ for $\alpha_i, \beta_i \in \mathbb{R}$ is given by (8) and $\gamma \in \mathbb{R}$. The s-space $\mathbf{S}^k = (\mathbb{R}^k, \oplus, \mathbb{R}, *)$ is called the *canonical s-space (of standard deviations)*.

2.2 Relations Between S-spaces and Vector Spaces

PROPOSITION 3. *Let* $(\mathcal{S}, +, \mathbb{R}, *)$ *be an s-space over* $\mathbb{R}$. *Then the system* $(\mathcal{S}, +, \mathbb{R}, \cdot)$ *where the operation* "$\cdot$": $\mathbb{R} \times \mathcal{S} \longrightarrow \mathcal{S}$ *is defined by*

$$\alpha \cdot c = \begin{cases} \sqrt{|\alpha|} * c, & \text{if } \alpha \geq 0; \\ \sqrt{|\alpha|} * (-c), & \text{if } \alpha < 0, \end{cases} \quad (12)$$

is a vector space over $\mathbb{R}$.

PROPOSITION 4. *Let* $(\mathcal{S}, +, \mathbb{R}, \cdot)$ *be a vector space over* $\mathbb{R}$. *The system* $(\mathcal{S}, +, \mathbb{R}, *)$, *where* "$*$" *is defined by*

$$\alpha * c = \alpha^2 \cdot c \quad (13)$$

is an s-space over $\mathbb{R}$.

Thus each of the two spaces $(\mathcal{S}, +, \mathbb{R}, *)$ and $(\mathcal{S}, +, \mathbb{R}, \cdot)$ can be obtained from the other one by a redefinition of the operation multiplication by scalars using (12), resp. (13).

Assume that $\mathcal{S} = (\mathcal{S}, +, \mathbb{R}, *)$ is an s-space over $\mathbb{R}$ and $(\mathcal{S}, +, \mathbb{R}, \cdot)$ is the associated vector space. From the vector

space $(\mathcal{S}, +, \mathbb{R}, \cdot)$ we can transfer vector space concepts, such as linear combination, linear dependence, basis etc., to the s-space $(\mathcal{S}, +, \mathbb{R}, *)$ [4]. Thus, it can be proved that any s-space over $\mathbb{R}$, with a basis of k elements, is isomorphic to $\mathbf{S}^k$.

Stochastic numbers can be defined as elements of the direct sum $\mathcal{V} \oplus \mathcal{S}$ of a vector space $\mathcal{V}$ (of mean values) and a s-space $\mathcal{S}$ (of standard deviations) both of same dimension k. Namely, let $\mathcal{V} = \mathbf{V}^k$ be a k-dimensional vector space with a basis $(v^{(1)}, ..., v^{(k)})$ and let $\mathcal{S} = \mathbf{S}^k$ be a k-dimensional s-space having a basis $(s^{(1)}, ..., s^{(k)})$. Then we say that $(v^{(1)}, ..., v^{(k)}; s^{(1)}, ..., s^{(k)})$ is a basis of the k-*dimensional* space $\mathbf{V}^k \oplus \mathbf{S}^k$. Such a setting allows us to consider numerical problems involving vectors and matrices, wherein the numeric variables have been substituted by stochastic ones. In the next section we consider such a problem.

3. LINEAR SYSTEMS WITH STOCHASTIC RIGHT-HAND SIDE

We consider a linear system $Ax = b$, such that A is a real $n \times n$-matrix and the right-hand side b is a vector of stochastic numbers. Then the solution x also consists of stochastic numbers, and, respectively, all arithmetic operations (additions and multiplications by scalars) in the expression Ax involve stochastic numbers; therefore we shall write $A * x$ instead of Ax.

Problem. Assume that $A = (\alpha_{ij})_{i,j=1}^n$, $\alpha_{ij} \in \mathbb{R}$, is a real $n \times n$-matrix, and $b = (b'; b'')$ is a n-tuple of (generalized) stochastic numbers, such that $b', b'' \in \mathbb{R}^n$, $b' = (b'_1, ..., b'_n)$, $b'' = (b''_1, ..., b''_n)$. We look for a (generalized) stochastic vector $x = (x'; x'')$, $x', x'' \in \mathbb{R}^n$, that is an n-tuple of stochastic numbers, such that $A * x = b$.

Solution. The i-th equation of the system $A * x = b$ reads $\alpha_{i1} * x_1 \oplus ... \oplus \alpha_{in} * x_n = b_i$. Obviously, $A * x = b$ reduces to a linear system $Ax' = b'$ for the vector $x' = (x'_1, ..., x'_n)$ of mean values and a system $A * x'' = b''$ for the standard deviations $x'' = (x''_1, ..., x''_n)$. If $A = (\alpha_{ij})$ is nonsingular, then $x' = A^{-1}b'$. We shall next concentrate on the solution of the system $A * x'' = b''$ for the standard deviations.

The i-th equation of the system $A * x'' = b''$ reads $\alpha_{i1} * x''_1 \oplus ... \oplus \alpha_{in} * x''_n = b''_i$. According to Proposition 2, this is equivalent to

$$\alpha_{i1}^2 \sigma(x''_1) x''^2_1 + ... + \alpha_{in}^2 \sigma(x''_n) x''^2_n = \sigma(b''_i) b''^2_i, \quad i = 1, ..., n,$$

minding that $\sigma(\alpha_{ij} * x''_j) = \sigma(x''_j)$.

Setting $\sigma(x''_i)(x''_i)^2 = y_i$, $\sigma(b''_i)(b''_i)^2 = c_i$, we obtain a linear $n \times n$ system $Dy = c$ for $y = (y_i)$, where $D = (\alpha_{ij}^2)$, $c = (c_i)$. If D is nonsingular we can solve the system $Dy = c$ for the vector y, and then obtain the standard deviation vector x'' by means of $x''_i = \sigma(y_i)\sqrt{|y_i|}$. Thus for the solution of the original problem it is necessary and sufficient that both matrices $A = (\alpha_{ij})$ and $D = (\alpha_{ij}^2)$ are nonsingular.

Summarizing, to solve $A * x = b$ the following steps are performed:

i) check the matrices $A = (\alpha_{ij})$ and $D = (\alpha_{ij}^2)$ for non-singularity;

ii) find the solution $x' = A^{-1}b'$ of the linear system $Ax' = b'$;

iii) find the solution $y = D^{-1}c$ of the linear system $Dy = c$, where $c = (c_i)$, $c_i = \sigma(b''_i)(b''^2_i)$. Compute $x''_i = \sigma(y_i)\sqrt{|y_i|}$;

iv) the solution of $A * x = b$ is $x = (x'; x'')$.

4. NUMERICAL EXPERIMENTS

4.1 Description of the experiments

Some numerical tests have been performed in order to compare the theoretical results with the numerical results obtained by means of the CESTAC method for imprecise stochastic data.

Let a be a real vector of size N and let b is a stochastic vector of same size N. According to the theory of the CES-TAC method a stochastic number and consequently each component b_i of b can be represented by a k-tuple of gaussian random values with known mean value m and standard deviation s. In the CADNA software [2], [3] the method is implemented with $k = 3$.

In the following experiments all the k-tuples for the components b_i have been generated with a gaussian generator with $m = 1$, $s = 0.001$.

To experiment the validity of the preceding theory and its concordance with the CESTAC method the tests have been performed as follows : For different sizes N the stochastic dot product $p = a.b$, i.e. the k components $p_i = \sum_{j=1}^N a_i b_{i,j}$ are computed, each component b_i being a k-tuple $(b_{i,1}, ..., b_{i,k})$ of random gaussian values as explained above.

Then the empiric mean value $\overline{p}$ and the empiric variance S^2 of the result are computed :

$$\overline{p} = \frac{1}{k} \sum_{i=1}^k p_i, \quad S^2 = \frac{1}{k-1} \sum_{i=1}^k (p_i - \overline{p})^2.$$

This provides samples of size k for the dot product p. The mean value and standard deviation of these samples respectively approximate the theoretical mean value μ and the theoretical standard deviation τ of p.

Moreover as p is a linear combination of gaussian variables it also follows a gaussian distribution. Hence $\frac{\sqrt{N-1}}{S}(\overline{p} - \mu)$ follows a Student law with a $k-1$ degree of freedom and $\frac{k S^2}{\tau^2}$ follows a χ^2 law with a $k-1$ degree of freedom. The difficulty of the theory being on the algebraic operations on variances it is this last property which is tested. Thus it is checked whether or not the theoretical variance τ^2 belongs to the confidence interval

$$[\frac{k S^2}{u_\alpha}; \frac{k S^2}{v_\alpha}]$$

where u_α and v_α are the values of the χ^2 distribution for a probability α and a $k-1$ degree of freedom. Here the values are taken for $\alpha = 0.95$.

4.2 Sum of N numbers

The vector a is defined as $a = (1, 1, ..., 1)$. In this case the dot product ab is equal to the sum of the stochastic components of b and thus the theoretical mean value is $\mu_1 = N$, and the theoretical standard deviation is $\tau_1 = s\sqrt{N} = 0.001\sqrt{N}$.

Table 1 reports the percentages of cases where the theoretical variance $\tau_1^2 = s N$ is inside the computed confidence interval. These percentages have been computed with 10000 runs.

N \ k	3	4	5	6	10	30
10	89.80	90.57	91.83	92.96	93.39	94.50
100	90.22	90.87	92.12	92.57	93.76	95.10
1000	88.99	91.16	92.22	92.67	93.37	93.70
10000	89.87	91.36	91.78	92.47	93.70	94.90

Table 1: Percentages of theoretical standard deviation τ_1^2 inside the confidence interval

4.3 Dot Product

We present the numerical results of two experiments.

Experiment 1. In this example a is a real vector with $(a_i = i, \ i = 1, ..., N)$, and b is a stochastic vector. All the samples have been generated with a gaussian generator $m = 1, s = 0.001$.

The theoretical variance of the dot product ab is $\tau_2^2 = s^2 \ N * (N + 1) * (2n + 1)/6$. Table 2 shows the percentages of theoretical variance τ_2^2 inside the confidence interval.

N \ k	3	4	5	6	10	30
10	89.51	90.85	92.43	92.16	93.04	94.70
100	90.15	90.88	92.46	92.89	93.05	94.30
1000	90.39	91.21	92.18	92.39	93.40	94.10
10000	89.77	90.85	91.29	92.82	93.41	93.50

Table 2: Percentages of theoretical standard deviation σ_2 insideside the confidence interval

Experiment 2. In this second example a is a real vector with $A(i) = 1/i, \ i = 1, ..., N$ and b is stochastic vector. All the samples have been generated with a gaussian generator $m = 1, s = 0.001$.

The theoretical variance is $\tau_3^2 = s^2 \sum_{i=0}^{n} 1/i^2$. Table 3 shows the percentages of theoretical variance τ_3^2 inside the confidence interval. Note: it is known that $\sum_{i=0}^{\infty} 1/i^2 = \pi^2/6$.

N \ k	3	4	5	6	10	30
10	90.68	92.39	93.13	93.52	94.64	94.50
100	89.83	91.06	91.76	92.81	93.68	96.00
1000	89.53	91.38	92.26	92.48	93.76	94.20
10000	89.82	91.26	91.93	92.71	93.72	94.50

Table 3: Percentages of theoretical variance τ_3^2 inside the confidence interval

Remark : These experiments on dot products show clearly that the values for the variances predicted by our theory are very close to the experimental ones. Anyhow one can see a very small bias as the experiments show that the theoretical values happen to be in the confidence interval with a probability which is rather .93 or .94 than .95 as it should be.

4.4 Solution of a linear system $A * x = b$

Let $A = \{a_{ij}\}$ be a real matrix such that $a_{ij} = i$, if $i = j$ else $a_{ij} = 10^{-|i-j|}$, $i, j = 1, .., N$. Assume that b is a stochastic vector such that the component b_i is generated with a gaussian generator with a mean value $b_i = \sum_{j=1}^{n} a_{ij}$ and a standard deviation equal to $1.e - 4$. With such kind of system, the solutions x_i are around 1.

The theoretical standard deviation is obtained according the method described in the previous section. We first compute matrix D from the matrix A. We solve first $y = D^{-1}c$, then we compute $x_i'' = \sigma(y_i)\sqrt{|y_i|}$. The values x_i'' are presented in Table 4.

Compo-nent	Theoretical standard deviations x''	Computed standard deviations
1	9.98e-05	10.4e-05
2	4.97e-05	4.06e-05
3	3.32e-05	3.21e-05
4	2.49e-05	2.02e-05
5	1.99e-05	1.81e-05
6	1.66e-05	1.50e-05
7	1.42e-05	1.54e-05
8	1.24e-05	1.02e-05
9	1.11e-05	0.778e-06
10	0.999e-05	0.806e-06

Table 4: Theoretical and computed standard deviations

To obtain the experimental results, we take 10 different vectors $b^{(k)}, k = 1, .., 10$, and solve the corresponding 10 system $A * x = b^{(k)}$. For each component of $b^{(k)}$, the standard deviation of the $n = 3$ samples is computed. In the end, the mean value of the standard deviations is computed for the $N = 10$ components and presented in Table 4.

As we can see in Table 4, the theoretical standard deviations and the computed values are very near.

5. CONCLUSION

In this work we briefly outline the algebraic theory of stochastic numbers related to the operations addition and multiplication by scalar and apply this theory for the solution of a linear algebraic problem.

The theoretic study of the properties of stochastic numbers allow us to obtain rigorous abstract definition of stochastic numbers with respect to the operations addition and multiplication by scalars. Our theory also allows us to solve algebraic problems with stochastic numbers. This gives us a possibility to compare algebraically obtained results with practical applications of stochastic numbers, such as the ones provided by the CESTAC method [2]. Such comparisons will give additional information related to the stochastic behaviour of random roundings in the course of numerical computations.

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