Abstract structures in stochastic arithmetic

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Abstract

In this work we study relevant ringoid structures in stochastic arithmetic involving multiplication. The study of such structures are needed in order to solve linear problems involving stochastic numbers in the matrix and not only in the right-hand side as it was formerly studied. This continues and extends a previous work where it has been shown that computation with generalized stochastic numbers as regard to addition and multiplication by scalars can be reduced to computation in familiar vector spaces. This allowed us to solve certain practical problems with stochastic numbers and to compare algebraically obtained results with practical applications of stochastic numbers. So this theoretical works aims at extending the results obtained with the vector space structure to corresponding operations in a ring-like structure.

Keywords: stochastic numbers, stochastic arithmetic, standard deviations, s-space, linear stochastic system, s-ring, CESTAC method.

1 Introduction

Stochastic numbers are gaussian random variables with a known mean value and a known standard deviation. Operations on stochastic numbers are used here as a model for operations on imprecise numbers, i.e. real numbers containing an unknown error which is supposed to be centred gaussian with a known standard deviation.

In practice, stochastic numbers are computed using the CESTAC method, which is a Monte-Carlo method consisting in performing each arithmetic operation several times using an arithmetic with a random rounding mode, see [2], [7], [8]. Some fundamental properties of stochastic numbers are considered in [3], [9].

This work is a part of a more general one, see [1], [5], [6], which consists in studying the algebraic structures induced by the operations on stochastic numbers in the same way as it has been done for intervals in order to provide a good algebraic understanding of the performance of the CESTAC method which has been up to now presented from a probabilistic point of view.

The mean values of the stochastic numbers satisfy the usual real arithmetic, whereas standard deviations are added and multiplied by scalars in a specific way. As regard to addition the system of standard deviations is an abelian monoid with cancellation law. This monoid can be embedded in an additive group and after a suitable extension of multiplication by scalars one obtains a so-called s-space, which is closely related to a vector space [1], [5]. This allows us to introduce in s-spaces concepts like linear combination, basis, dimension etc. Thus, in theory, computations in s-spaces are reduced to computations in vector spaces. This enables to find explicit expressions for the solution of certain algebraic problems involving stochastic numbers.

In Sections 2 we briefly present the main results of the theory of s-spaces as regard to the arithmetic operations for addition and multiplication by scalars needed for the purposes of this study; for a detailed presentation of the theory, see [5]. As an example we consider the algebraic solution of linear systems of equations which right-hand sides involve stochastic numbers. In Section 3 we study relevant ringoid structures in stochastic arithmetic involving multiplication.

2 Stochastic Arithmetic: Addition and Multiplication by Scalars

By $\mathbb{R}$ we denote the set of reals; the linearly ordered field of reals is denoted $\mathbb{R}_D = (\mathbb{R}, +, \cdot, \leq)$. For any integer $n \geq 1$ we denote by $\mathbb{R}^n$ the set of all n-tuples $(\alpha_1, \alpha_2, ..., \alpha_n)$, where $\alpha_i \in \mathbb{R}$. The set $\mathbb{R}^n$ forms a vector space under the familiar operations of addition and multiplication by scalars denoted by $\mathbb{R}^n = (\mathbb{R}^n, +, \mathbb{R}_D, \cdot)$, $n \geq 1$. By $\mathbb{R}^+$ we denote the set of nonnegative real numbers.

A *stochastic number* $X = (m; s)$ is a gaussian random variable with mean value $m \in \mathbb{R}$ and (nonnegative) standard deviation $s \in \mathbb{R}^+$. The set of all stochastic numbers is $\mathbb{S} = \{(m; s) \mid m \in \mathbb{R}, s \in \mathbb{R}^+\}$.

The arithmetic for stochastic numbers. Let $X_1 = (m_1; s_1)$, $X_2 = (m_2; s_2) \in \mathbb{S}$. Addition and multiplication by scalars are defined by:

$$X_1 + X_2 = (m_1; s_1) + (m_2; s_2) \stackrel{def}{=} \left(m_1 + m_2;\ \sqrt{s_1^2 + s_2^2}\right),$$
$$\gamma * X = \gamma * (m; s) \stackrel{def}{=} \left(\gamma m;\ |\gamma| s\right),\ \gamma \in \mathbb{R}_D.$$

A stochastic number of the form $(0; s)$, $s \in \mathbb{R}^+$, is called *symmetric* or *centred*. If X_1, X_2 are symmetric stochastic numbers, then $X_1 + X_2$ and $\lambda * X_1$, $\lambda \in \mathbb{R}_D$, are also symmetric stochastic numbers. Thus there is a 1–1 correspondence between the set of symmetric stochastic numbers and the set $\mathbb{R}^+$. We shall use special symbols "$\oplus$", "$*$" for the arithmetic operations over standard deviations, as these operations are different from the corresponding ones for numbers. The operations "$\oplus$", "$*$" induce a special arithmetic on the set $\mathbb{R}^+$. Consider the system $(\mathbb{R}^+, \oplus, \mathbb{R}_D, *)$, where:

$$\alpha \oplus \beta = \sqrt{\alpha^2 + \beta^2},\ \alpha, \beta \in \mathbb{R}^+, \quad (1)$$
$$\gamma * \delta = |\gamma| \delta,\ \gamma \in \mathbb{R}_D,\ \delta \in \mathbb{R}^+. \quad (2)$$

Proposition 1. *[5] The system $(\mathbb{R}^+, \oplus, \mathbb{R}_D, *)$ is an abelian additive monoid with cancellation, such that for $s, t \in \mathbb{R}^+$, $\alpha, \beta \in \mathbb{R}_D$:*

$$\alpha * (s \oplus t) = \alpha * s \oplus \alpha * t, \quad (3)$$
$$\alpha * (\beta * s) = (\alpha\beta) * s, \quad (4)$$
$$1 * s = s, \quad (5)$$
$$(-1) * s = s, \quad (6)$$
$$\sqrt{\alpha^2 + \beta^2} * s = \alpha * s \oplus \beta * s,\ \alpha, \beta \geq 0. \quad (7)$$

A system satisfying the conditions of Proposition 1 is called an s-space of monoid structure.

2.1 The Group System of Standard Deviations

For $\alpha \in \mathbb{R}$ denote $\sigma(\alpha) = \{+,\ \text{if}\ \alpha \geq 0;\ -,\ \text{if}\ \alpha < 0\}$. We extend the operation addition "$\oplus$" for all $\alpha, \beta \in \mathbb{R}$, admitting thus negative reals, corresponding to *improper* standard deviations:

$$\alpha \oplus \beta \stackrel{def}{=} \sigma(\alpha + \beta)\sqrt{|\sigma(\alpha)\alpha^2 + \sigma(\beta)\beta^2|}. \quad (8)$$

Note that $\sigma(\alpha + \beta) = \sigma(\sigma(\alpha)\alpha^2 + \sigma(\beta)\beta^2) = \sigma(\alpha \oplus \beta)$ for $\alpha, \beta \in \mathbb{R}$.

Using (8) we embed isomorphically the monoid $(\mathbb{R}^+, \oplus)$ into the system $(\mathbb{R}, \oplus)$, which is an abelian group with null 0 and opposite element $\mathrm{opp}(\alpha) = -\alpha$, i. e. $\alpha \oplus (-\alpha) = 0$. Indeed, (8) implies $\alpha \oplus (-\alpha) = \sigma(\alpha - \alpha)\sqrt{|\sigma(\alpha)\alpha^2 - \sigma(\alpha)\alpha^2|} = \sigma(0)\sqrt{0} = 0$. Here are some examples of addition in the system $(\mathbb{R}, \oplus)$: $1 \oplus 1 = \sqrt{2}$, $1 \oplus 2 = \sqrt{5}$, $3 \oplus 4 = 5$, $4 \oplus (-3) = \sqrt{7}$, $3 \oplus (-4) = -\sqrt{7}$, $5 \oplus (-4) = 3$,

$4 \oplus (-5) = -3$, $(-3) \oplus (-4) = -5$, $1 \oplus 2 \oplus 3 = \sqrt{14}$, $1 \oplus 2 \oplus (-3) = -2$.

Using (8) we obtain for $n \geq 2$

$$\alpha_1 \oplus ... \oplus \alpha_n = \\ \sigma(\alpha_1 \oplus ... \oplus \alpha_n) \sqrt{|\sigma(\alpha_1)\alpha_1^2 + ... + \sigma(\alpha_n)\alpha_n^2|}.$$

Proposition 2. *The equation $\alpha_1 \oplus \alpha_2 \oplus ... \oplus \alpha_n = \beta$ is equivalent to $\sigma(\alpha_1)\alpha_1^2 + ... + \sigma(\alpha_n)\alpha_n^2 = \sigma(\beta)\beta^2$.*

Multiplication by scalars is naturally extended on the set $\mathbb{R}$ of generalized standard deviations by: $\gamma * s = |\gamma| s$, $s \in \mathbb{R}$. Multiplication by -1 (negation) is $(-1) * s = |-1| s = s$, $s \in \mathbb{R}$. To avoid confusion we shall write the scalars always to the left side of the standard deviation. Under this convention we have, e. g. $(-2) * 2 = 4$, whereas $2 * (-2) = -4$. Note that if s is a standard deviation, then we have $\gamma * s = (-\gamma) * s$ for any scalar $\gamma \in \mathbb{R}_D$.

It is easy to check that relations (3)–(7) hold true for generalized standard deviations. This justifies the following definition:

Definition 1. *A system $(\mathcal{S}, \oplus, \mathbb{R}_D, *)$, such that:*

i) $(\mathcal{S}, \oplus)$ is an abelian additive group, and

ii) for any $s, t \in \mathcal{S}$ and $\alpha, \beta \in \mathbb{R}_D$ relations (3)–(7) hold,

is called an s-space over $\mathbb{R}_D$ (of group structure).

The canonical s-space. For any integer $k \geq 1$ the set $\mathcal{S} = \mathbb{R}^k$ of all k-tuples $(\alpha_1, \alpha_2, ..., \alpha_k)$ forms an s-space over $\mathbb{R}_D$ under the following operations

$$(\alpha_1, ..., \alpha_k) \oplus (\beta_1, ..., \beta_k) = \\ (\alpha_1 \oplus \beta_1, ..., \alpha_k \oplus \beta_k), \\ \gamma * (\alpha_1, \alpha_2, ..., \alpha_k) = (|\gamma|\alpha_1, ..., |\gamma|\alpha_k),$$

where $\alpha_i \oplus \beta_i$ for $\alpha_i, \beta_i \in \mathbb{R}$ is given by (8) and $\gamma \in \mathbb{R}_D$. The s-space $\mathbb{S}^k = (\mathbb{R}^k, \oplus, \mathbb{R}, *)$ is called the *canonical s-space (of standard deviations)*. We identify the elements of $\mathbb{S}^k$ with the generalized symmetric stochastic numbers.

2.2 S-spaces and Vector Spaces

Proposition 3. *Let $(\mathcal{S}, +, \mathbb{R}_D, *)$ be an s-space over $\mathbb{R}_D$. Then $(\mathcal{S}, +, \mathbb{R}_D, \cdot)$ is a vector space over $\mathbb{R}_D$, iff the operation "$\cdot$": $\mathbb{R}_D \times \mathcal{S} \longrightarrow \mathcal{S}$ is defined by*

$$\alpha \cdot c = \begin{cases} \sqrt{|\alpha|} * c, & \text{if } \alpha \geq 0; \\ \sqrt{|\alpha|} * (-c), & \text{if } \alpha < 0. \end{cases}$$

Proposition 4. *Let $(\mathcal{S}, +, \mathbb{R}_D, \cdot)$ be a vector space over $\mathbb{R}_D$. The system $(\mathcal{S}, +, \mathbb{R}_D, *)$, where "$*$" is defined by $\alpha * c = \alpha^2 \cdot c$ is an s-space over $\mathbb{R}_D$.*

Thus each of the two spaces $(\mathcal{S}, +, \mathbb{R}_D, *)$ and $(\mathcal{S}, +, \mathbb{R}_D, \cdot)$ can be obtained from the other by a redefinition of the operation multiplication by scalars using Proposition 3, resp. Proposition 4.

Assume that $\mathcal{S} = (\mathcal{S}, +, \mathbb{R}_D, *)$ is an s-space over $\mathbb{R}_D$ and $(\mathcal{S}, +, \mathbb{R}_D, \cdot)$ is the associated vector space. From the vector space $(\mathcal{S}, +, \mathbb{R}_D, \cdot)$ we can transfer vector space concepts, such as linear combination, linear dependence, basis etc., to the s-space $(\mathcal{S}, +, \mathbb{R}_D, *)$ [5]. Thus, it can be proved that any s-space over $\mathbb{R}_D$, with a basis of k elements, is isomorphic to $\mathbb{S}^k$.

Generalized stochastic numbers can be defined as elements of the direct sum $\mathbb{R}^k \oplus \mathbb{S}^k$ of a vector space $\mathbb{R}^k$ (of mean values) and a s-space $\mathbb{S}^k$ (of standard deviations) both of same dimension k. Such a setting allows us to consider numerical problems involving vectors and matrices, wherein the numeric variables have been substituted by stochastic ones. In the next subsection we consider such a problem.

2.3 Linear Systems with Stochastic Right-hand Side

We consider a linear system $Ax = b$, such that A is a real $n \times n$-matrix and the right-hand side b is a vector of stochastic numbers. Then the solution x also consists of stochastic numbers, and, respectively, all arithmetic operations (additions and multiplications by scalars) in the expression Ax involve stochastic numbers; therefore we shall write $A * x$ instead of Ax.

Problem. Assume that $A = (\alpha_{ij})_{i,j=1}^{n}$, $\alpha_{ij} \in \mathbb{R}$, is a real $n \times n$-matrix, and $b = (b'; b'')$ is a n-tuple of (generalized) stochastic numbers, such that $b', b'' \in \mathbb{R}^n$, $b' = (b'_1, ..., b'_n)$, $b'' = (b''_1, ..., b''_n)$. We look for a (generalized) stochastic vector $x = (x'; x'')$, $x', x'' \in \mathbb{R}^n$, that is an n-tuple of stochastic numbers, such that $A * x = b$.

Solution. The i-th equation of the system $A * x = b$ reads $\alpha_{i1} * x_1 \oplus ... \oplus \alpha_{in} * x_n = b_i$. Obviously, $A * x = b$ reduces to a linear system $Ax' = b'$ for the vector $x' = (x'_1, ..., x'_n)$ of mean values and a system $A * x'' = b''$ for the standard deviations $x'' = (x''_1, ..., x''_n)$. If $A = (\alpha_{ij})$ is nonsingular, then the solution for the mean-values is straightforward and can be obtained by any method for solving a real linear system. We shall next concentrate on the solution of the system $A * x'' = b''$ for the standard deviations which is the one that makes problem because of the corresponding peculiar addition and multiplication by scalars.

The i-th equation of the system $A * x'' = b''$ reads $\alpha_{i1} * x''_1 \oplus ... \oplus \alpha_{in} * x''_n = b''_i$. According to Proposition 2, this is equivalent to

$$\alpha_{i1}^2 \sigma(x''_1) x''^2_1 + ... + \alpha_{in}^2 \sigma(x''_n) x''^2_n = \sigma(b''_i) b''^2_i,$$

minding that $\sigma(\alpha_{ij} * x''_j) = \sigma(x''_j)$.

Setting $\sigma(x''_i)(x''_i)^2 = y_i$, $\sigma(b''_i)(b''_i)^2 = c_i$, we obtain a linear $n \times n$ system $Dy = c$ for $y = (y_i)$, where $D = (\alpha_{ij}^2)$, $c = (c_i)$. If D is nonsingular we can solve the system $Dy = c$ for the vector y, $y = D^{-1}c$, and then obtain the standard deviation vector x'' by means of $x''_i = \sigma(y_i)\sqrt{|y_i|}$. Thus for the solution of the original problem it is necessary and sufficient that both matrices $A = (\alpha_{ij})$ and $D = (\alpha_{ij}^2)$ are nonsingular.

3 Stochastic Arithmetic: Multiplication and Inclusion

In the previous section we discussed the algebraic properties of stochastic numbers (or vectors) related to the operations addition and multiplication by scalars. Now we turn our attention to one more operation—multiplication—and the inclusion relation. The aim is here to develop a theory extending the previous one and in which, full linear systems in which both left hand side and right hand side contain stochastic numbers. Thus the multiplication of two stochastic numbers must be defined and some structures close to rings and then to fields must be constructed. In this work consider the one-dimensional case is only considered.

Two stochastic numbers $X_1 = (m_1; s_1)$, $X_2 = (m_2; s_2)$, $s_1, s_2 \geq 0$, are multiplied according to:

$$X_1 * X_2 \stackrel{def}{=} \left(m_1 m_2;\ \sqrt{m_1^2 s_2^2 + m_2^2 s_1^2 + s_2^2 s_1^2} \right). \qquad (9)$$

The inclusion relation "$\subseteq$" between two stochastic numbers is defined by

$$X_1 \subseteq X_2 \Longleftrightarrow | m_2 - m_1 | \leq s_2 - s_1. \qquad (10)$$

We now proceed as in the previous section: we extend (9), (10) for generalized (proper and improper) stochastic numbers starting from the set $\mathbb{S}$ of symmetric generalized stochastic numbers. In our construction we shall require that multiplication is inclusion isotone, that is: $X_1 \subseteq X_2$ implies $X_1 * Y \subseteq X_2 * Y$ for any stochastic numbers X_1, X_2, Y. Note that the latter property is satisfied for addition, that is: $X_1 \subseteq X_2$ implies $X_1 + Y \subseteq X_2 + Y$. We define multiplication in $\mathbb{S}$ following ideas from [4]. Consider the following operation for $(0; a''), (0; b'') \in \mathbb{S}$:

$$(0; a'') * (0; b'') = (0;\ a'' * b''), \qquad (11)$$

where $a'' * b''$ is given by:

$$a * b = \begin{cases} ab, & \text{if } a \geq 0,\ b \geq 0, \\ -ab, & \text{if } a \leq 0,\ b < 0, \\ 0, & \text{if } a > 0,\ b < 0 \\ & \quad \text{or } a < 0,\ b > 0. \end{cases} \qquad (12)$$

For example: $(0; 2) * (0; 3) = (0; 6)$, $(0; -2) * (0; -3) = (0; -6)$, $(0; -2) * (0; 3) = (0; 0)$. Inclusion is extended in $\mathbb{S}$ by means of $(0; a'') \subseteq (0; b'') \Longleftrightarrow a'' \leq b''$ after dropping the restriction $a'' \geq 0$, $b'' \geq 0$. For example:

$(0;-3) \subseteq (0;-1) \subseteq (0;0) \subseteq (0;1)$. This order corresponds to the natural order in the set of reals. Is is easy to see that the inclusion isotonicity property for multiplication is satisfied.

Using this idea it is natural to define multiplication in the set of generalized stochastic numbers by means of

$$a * b = \begin{cases} (a'b'; |a'|b'' + |b'|a'' + \tilde{a}''\tilde{b}''), \\ \text{ if } \kappa(a) \le 1,\ \kappa(b) \le 1; \\ (b' + \sigma(b')\sigma(a'')b'') * (a'; a''), \\ \text{ if } \kappa(a) > 1 \ge \kappa(b) \text{ or} \\ \kappa(a) \ge \kappa(b) > 1, a''b'' \ge 0; \\ 0,\ \text{if } \kappa(a) \ge 1,\ \kappa(b) \ge 1, a''b'' \le 0, \end{cases}$$

wherein the functional κ is given for $a = (a'; a''))$ by: $\kappa(a) = |a''|/|a'|,\ a' \neq 0$.

We note that multiplication by scalars, as defined in the previous section, is compatible with the above operation for multiplication, which allows us to use same notation for both operations.

In what follows we discuss the system $(\mathbb{R} \oplus \mathbb{S}, +, *, \subseteq)$ in axiomatic manner starting from the concept of s-ring.

3.1 Axiomatic definition of s-ring

Definition 2. *An* s-ring *is a system* $(\mathcal{S}, \oplus, *, \subseteq)$*, such that:*

A1. $(\mathcal{S}, \subseteq)$ *is an ordered set (a poset);*

A2. $(\mathcal{S}, \oplus, \subseteq)$ *is an isotone abelian group;*

A3. Multiplication in $(\mathcal{S}, *, \subseteq)$ *is isotone;*

A4. Multiplication "$*$" in $(\mathcal{S}, \oplus, *, \subseteq)$ *is distributive over a sum of any two elements that are* $\supseteq 0$.

If, in addition, the multiplication "$*$" is commutative, we shall call the s-ring *commutative.*

Recall that the ordered field of reals $\mathbb{R}_D = (\mathbb{R}, +, \cdot, \le)$ is a (linearly) ordered ring. Indeed, $\mathbb{R}_D$ satisfies assumptions A1 and A2. Concerning assumption A3, $(\mathbb{R}, \cdot, \le)$ is a commutative semigroup, which is not isotone, that is $a \le b \Longrightarrow ac \le bc$ does not generally hold (for all a, b, c). As regard to assumption A4, the ring $(\mathbb{R}, +, \cdot)$ is distributive over the sum of any two elements, whereas $(\mathcal{S}, \oplus, *)$ is not.

For more clarity let us formulate the assumptions of a s-ring in detail using symbolic notation. The null in the additive group $(\mathcal{S}, \oplus)$ is denoted as usually by 0; the opposite element to $a \in \mathcal{S}$ is denoted $\mathrm{opp}(a)$. Using these notation, according to Definition 2, a system $(\mathcal{S}, \oplus, *, \subseteq)$ is a *s-ring* if the following 10 axioms hold true:

A1. $(\mathcal{S}, \subseteq)$ is ordered, i. e. for all $a, b, c \in \mathcal{S}$:

- A1.1. $a \subseteq a$,
- A1.2. $a \subseteq b,\ b \subseteq a \Longrightarrow a = b$,
- A1.3. $a \subseteq b,\ b \subseteq c \Longrightarrow a \subseteq c$;

A2. $(\mathcal{S}, \oplus, \subseteq)$ is an isotone abelian group with null 0 and opposite $\mathrm{opp}(a)$, i. e. for all $a, b, c \in \mathcal{S}$:

- A2.1. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$,
- A2.2. $a \oplus b = b \oplus a$,
- A2.3. $a \oplus 0 = a$,
- A2.4. $a \oplus \mathrm{opp}(a) = 0$,
- A2.5. $a \subseteq b \Longrightarrow a \oplus c \subseteq b \oplus c$;

A3. Multiplication in $(\mathcal{S}, *, \subseteq)$ is isotone, i. e. $a \subseteq b \Longrightarrow a * c \subseteq b * c$ for all $a, b, c \in \mathcal{S}$;

A4. For all $a, b, c \in \mathcal{S}$ such that $0 \subseteq a, b$, $(\mathcal{S}, \oplus, *, \subseteq)$ satisfies $(a \oplus b) * c = a * c \oplus b * c$.

Now several definitions follow analogous to corresponding definitions related to familiar rings.

Definition 3. *An s-ring* $\mathcal{S}$ *is an* s-ring with identity *if there is an element* e *in* $\mathcal{S}$ *such that* $a * e = e * a = a$ *for all* $a \in \mathcal{S}$.

If there is an identity, it is clearly unique, and will be denoted, as usually, by 1. Further, as with familiar rings, if the s-ring is not the null one (that is consists only of the null element 0), then $1 \neq 0$.

Definition 4. *Let $\mathcal{S}$ be a* s-ring. *If $a \neq 0, b \neq 0$ are elements of $\mathcal{S}$ such that $a * b = 0$, then a is called left zero divisor and b is called right zero divisor.*

A s-ring is linearly ordered if for any a, b either $a \subseteq b$ or $b \subseteq a$ holds true. In a linearly ordered s-ring denote by $\mathcal{S}_+ = \{a \in \mathcal{S} \mid a \supseteq 0\}$, $\mathcal{S}_- = \mathcal{S} \backslash \mathcal{S}_+$, the sets of *proper*, resp. *improper* elements of $\mathcal{S}$; the set $\mathcal{S}_+ \setminus \{0\}$ comprises the *strictly proper* elements of $\mathcal{S}$.

We introduce a function $\tau : \mathcal{S} \longrightarrow \Lambda = \{+, -\}$ (type, direction, orientation of an element of $\mathcal{S}$) by

$$\tau(a) = \begin{cases} +, & \text{if } a \in \mathcal{S}_+, \\ -, & \text{if } a \in \mathcal{S}_-. \end{cases} \tag{13}$$

Clearly $\tau(a) = \tau(b)$, if $a, b \in \mathcal{S}_+$ or $a, b \in \mathcal{S}_-$". (The function τ is similar to the function σ, but τ is defined in an s-ring whereas σ is defined in a ring—the field of reals.)

Theorem 1. *(Quasidistributive law in an s-ring) For any a, b, c from an s-ring we have:*

$$(a \oplus b) * c_{\tau(a \oplus b)} = a * c_{\tau(a)} \oplus b * c_{\tau(b)}. \tag{14}$$

3.2 Relation between a linear ordered (l.o.) ring and a s-ring

Formula (12) defines multiplication "$*$" in a l. o. ring by the familiar linear multiplication. Similarly, we can define linear multiplication in a l. o. s-ring

$$a \cdot b = a_{\tau(b)} * b_{\tau(a)}, \tag{15}$$

where τ is defined by (13).

Theorem 2. *Every l. o. ring (l. o. field) generates via (12) a unique (up to isomorphism) s-ring and vice versa, every s-ring induces via (15) a unique ring.*

Theorem 2 shows that results from a field can be re-formulated as results in the induced s-rings by means of (15) and vice versa, results from an s-ring can be reformulated as results in the induced field by means of (12).

Definition 5. *An* s-algebra *is a l. o. s-ring, which is an l. o. quasivector space over the l. o. real field $\mathbb{R}_D$, and multiplications in both spaces are compatible.*

3.3 Spaces of Stochastic Numbers

Theorem 3. *The system $\mathbb{S} = (\mathbb{R}, +, *, \mathbb{R}_D, *', \subseteq)$ with multiplication "$*'$" defined by (12) and multiplication by scalars $\gamma * a = |\gamma| a$, is an s-algebra.*

This is the reason to use same notation "$*$" for the two multiplications. To avoid confusion, we place the scalar always to the left of the stochastic number.

Consider the direct sum $\mathbb{R}_D \oplus \mathbb{S}$. The element $a = (a'; a'') \in \mathbb{R}_D \oplus \mathbb{S}$ is a *(generalized) stochastic number.* Clearly, $(a'; 0) \in \mathbb{R}_D$ and $(0; a'') \in \mathbb{S}$. Addition of stochastic numbers is:

$$a \oplus b = (a' + b'; a'' \oplus b'') = (a' + b'; \sqrt{a''^2 \oplus b''^2}).$$

The opposite (inverse additive) of $a = (a', a'')$ is:

$$\mathrm{opp}(a'; a'') = \neg(a'; a'')_- = (-a'; a''_-) = (-a'; -a'').$$

In $\mathbb{R}_D \oplus \mathbb{S}$ we have multiplication by scalars $\gamma \in \mathbb{R}_D$:

$$\begin{aligned} \gamma * (a'; a'') &= (\gamma * a'; \gamma * a'') \\ &= (\gamma \cdot a'; \gamma * a'') = (\gamma a'; |\gamma| a''). \end{aligned}$$

Note that $(-1) * (a'; a'') = \neg(a'; a'') = (-a'; a'') \neq \mathrm{opp}(a'; a'')$.

Theorem 4. *The system $(\mathbb{R}_D \oplus \mathbb{S}, +, \mathbb{R}_D, *)$ is a quasivector space.*

We have $(-1) * (b'; b'') = (-b'; b'')$. We shall further denote $(-1) * a$ by $\neg a$ and call the operation $\neg a$ negation $\neg (a'; a'') = (-a'; a'')$.

We denote the composition of "opp" and "neg" in $\mathbb{R}_D \oplus \mathbb{S}$ as *conjugation (dual)*, symbolically $a_- = \mathrm{opp}(\neg a) = \neg(\mathrm{opp}(a))$. Note that in $\mathbb{S}$ negation is identity, $\neg(0; a'') = (0; a'')$, and we have $a_- = \mathrm{opp}(a)$ for $a \in \mathbb{S}$, that is $(0; a'')_- = \mathrm{opp}(0; a'') = (0; -a''_-) = (0; -a'')$.

On the other side, in $\mathbb{R}_D$ we have $(a'; 0)_- = (a'; 0)$, so that conjugation and identity coincide in $\mathbb{R}_D$. In general we have: $a_- = (a'; a'')_- = (-1) * \mathrm{opp}(a) = (a'; a''_-) = (a'; -a'')$.

We define *multiplication* in $\mathbb{R}_D \oplus \mathbb{S}$ by:

$$\begin{aligned}&(a';a'')*(b';b'')\\ =&\ (a'\cdot b';\ a'*b''+b'*a''+a''*b'')\\ =&\ (a'b';\ |a'|b''+|b'|a''+a''*b''),\end{aligned}$$

using the same notation "$*$" as in $\mathbb{S}$. From the special cases:

$$\begin{aligned}(a';0)*(b';0) &= (a'b';\ 0),\\ (0;a'')*(0;b'') &= (0;\ a''*b''),\end{aligned}$$

we conclude that (3.3) extends the multiplications from $\mathbb{R}_D$ and $\mathbb{S}$ into the direct sum $\mathbb{R}_D \oplus \mathbb{S}$. We also have

$$(a';0)*(b';b'') = (a'b';\ a'*b'') = (a'b';\ |a'|b''),$$

showing that a multiplier of the form $(a';0)$ acts like a scalar in multiplication by scalars. Hence elements of the form $(a';0)$ can be identified with scalars $a' \in \mathbb{R}_D$. In particular, we have $(-1;0)*(b';b'') = (-b';\ b'')$.

Commutativity. It is immediately seen that multiplication in $\mathbb{R}_D \oplus \mathbb{S}$ is commutative.

Identity. It is immediately seen that $(1;0)$ is an identity of $(\mathbb{R}_D \oplus \mathbb{S}, *)$: $(a';a'')*(1;0) = (a';a'')$.

Reciprocal. The solution x of the equation $a*x = 1$ for $a' \neq 0$ is $x = a^{-1} = (1/a'; -a''/|a'|^2)$. We say that a^{-1} is the reciprocal of a.

It is also shown easily that the multiplication by scalars and inner multiplication in $\mathbb{R}_D \oplus \mathbb{S}$ are compatible. In addition, a quasi-distributive relation of the form (14) can be proved. Inclusion is defined in $\mathbb{R}_D \oplus \mathbb{S}$ by means of (10). Inclusion monotonicity of multiplication in $\mathbb{R}_D \oplus \mathbb{S}$ holds as well. Due to the presence of unity division is introduced in $\mathbb{R}_D \oplus \mathbb{S}$.

Proposition 5. *The system* $(\mathbb{R}_D \oplus \mathbb{S}, \oplus, *, \subseteq)$ *is a s-ring.*

4 Conclusion

The theoretic study of the properties of stochastic numbers allow us to obtain rigorous abstract definition of stochastic numbers with respect to the operations addition, inner multiplication and multiplication by scalars. This allows us to solve certain algebraic problems with stochastic numbers. This gives us a possibility to compare algebraically obtained results with practical applications of stochastic numbers, such as the ones provided by the CESTAC method [2]. Such comparisons will give additional information related to the stochastic behaviour of random roundings in the course of numerical computations. In this work we study relevant ringoid structures in stochastic arithmetic involving (inner) multiplication. It is hoped that the study of such structures will allow us to solve linear problems involving stochastic numbers in the matrix (not only in the right-hand sides).

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