Numerical Computations with Hausdorff Continuous Functions

Roumen Anguelov[1] and Svetoslav Markov[2]

[1] University of Pretoria
Pretoria 0002, South Africa
roumen.anguelov@up.ac.za
[2] Institute of Mathematics and Informatics, BAS
"Acad. G. Bonchev" st., block 8, 1113 Sofia, Bulgaria
smarkov@bio.bas.bg

1 Introduction

Hausdorff continuous (H-continuous) functions appear naturally in many areas of mathematics like Approximation Theory [11], Real Analysis [1], [8], Interval Analysis, [2], etc. From numerical point of view it is significant that the solutions of large classes of nonlinear partial differential equations can be assimilated through H-continuous functions [7]. As a particular case the discontinuous viscosity solutions are also better represented through Hausdorff continuous functions [6]. Hence the need to develop numerical procedures for computations with H-continuous functions. It was shown recently, that the operations addition and multiplication by scalars of the usual continuous functions on $\Omega \subseteq \mathbb{R}^n$ can be extended to H-continuous functions in such a way that the set $\mathbb{H}(\Omega)$ of all Hausdorff continuous functions is a linear space [4]. In fact$\mathbb{H}(\Omega)$ is the largest linear space involving interval functions. Furthermore, multiplication can also be extended [5], so that $\mathbb{H}(\Omega)$ is a commutative algebra. Approximation of $\mathbb{H}(\Omega)$ by a subspace were discussed in [3]. In the present paper we consider the numerical computations with H-continuous functions using ultra-arithmetical approach [9], namely, by constructing a functoid of H-continuous functions. For simplicity we consider $\Omega \subseteq \mathbb{R}$. In the next section we recall the definition of the algebraic operations on $\mathbb{H}(\Omega)$. The concept of functoid is defined in Section 3. In Section 4 we construct a functoid comprising a finite dimensional subspace of $\mathbb{H}(\Omega)$ with a Fourier base extended by a set of H-continuous functions. Application of the functoid to the numerical solution of the wave equation is discussed in Section 5.

2 The algebra of H-continuous functions

The real line is denoted by $\mathbb{R}$ and the set of all finite real intervals by $\mathbb{IR} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R},\ \underline{a} \le \overline{a}\}$. Given an interval $a = [\underline{a}, \overline{a}] \in \mathbb{IR}$, $w(a) = \overline{a} - \underline{a}$ is the width of a. An interval a is called proper interval, if $w(a) > 0$ and point interval, if $w(a) = 0$. Identifying $a \in \mathbb{R}$ with the point interval $[a, a] \in \mathbb{IR}$, we consider $\mathbb{R}$ as a subset of $\mathbb{IR}$. Let $\Omega \subseteq \mathbb{R}$ be open. We recall [11] that an interval function $f : \Omega \to \mathbb{IR}$ is S-continuous if its graph is a closed subset of $\Omega \times \mathbb{R}$. An interval function $f : \Omega \to \mathbb{IR}$ is Hausdorff continuous (H-continuous) if it is an S-continuous function which is minimal with respect to inclusion, that is, if $\varphi : \Omega \to \mathbb{IR}$ is an S-continuous function and $\varphi \subseteq f$ implies $\varphi = f$. Here inclusion is understood pointwise. We denote by $\mathbb{H}(\Omega)$ the set of H-continuous functions on Ω. The following theorem states an essential property of the continuous functions which is preserved by the H-continuity [1].

Theorem 1. *Let $f, g \in \mathbb{H}(\Omega)$. If there exists a dense subset D of Ω such that $f(x) = g(x)$, $x \in D$, then $f(x) = g(x)$, $x \in \Omega$.*

H-continuous functions are also similar to usual continuous real functions in that they assume point values on a residual subset of Ω. More precisely, it is shown in [1] that for every $f \in \mathbb{H}(\Omega)$ the set $W_f = \{x \in \Omega : w(f(x)) > 0\}$ is of first Baire category and f is continuous on $\Omega \setminus W_f$. Since a finite or countable union of sets of first Baire category is also a set of first Baire category we have:

Theorem 2. *Let $\mathcal{F}$ be a finite or countable set of H-continuous functions. Then the set $D_{\mathcal{F}} = \{x \in \Omega : w(f(x)) = 0,\ f \in \mathcal{F}\} = \Omega \setminus \bigcup_{f\in\mathcal{F}} W_f$ is dense in Ω and all functions $f \in \mathcal{F}$ are continuous on $D_{\mathcal{F}}$.*

For every S-continuous function g we denote by $[g]$ the set of H-continuous functions contained in g, that is,

$$[g] = \{f \in \mathbb{H}(\Omega) : f \subseteq g\}.$$

Identifying $\{f\}$ with f we have $[f] = f$ whenever f is H-continuous. The S-continuous functions g such that the set $[g]$ is a singleton, that is, it contains only one function, play an important role in the sequel. In analogy with the H-continuous functions, which are minimal S-continuous functions, we call these functions quasi-minimal. The following characterization of the quasi-minimal S-continuous functions is an easy consequence of Theorem 1.

Theorem 3. *If the function f S-continuous on Ω and assumes point values on a dense subset of Ω, then f is a quasi-minimal S-continuous function.*

The familiar operations of addition, multiplication by scalars and multiplication on the set of real intervals are defined for $[\underline{a}, \overline{a}], [\underline{b}, \overline{b}] \in \mathbb{IR}$ and $\alpha \in \mathbb{R}$ as follows:

$$[\underline{a}, \overline{a}] + [\underline{b}, \overline{b}] = \{a + b : a \in [\underline{a}, \overline{a}], b \in [\underline{b}, \overline{b}]\} = [\underline{a} + \underline{b}, \overline{a} + \overline{b}],$$
$$\alpha \cdot [\underline{a}, \overline{a}] = \{\alpha a : a \in [\underline{a}, \overline{a}]\} = [\min\{\alpha\underline{a}, \alpha\overline{a}\}, \max\{\alpha\underline{a}, \alpha\overline{a}\}],$$
$$[\underline{a}, \overline{a}] \times [\underline{b}, \overline{b}] = \{ab : a \in [\underline{a}, \overline{a}], b \in [\underline{b}, \overline{b}]\} = [\min\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}, \max\{\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}\}].$$

Pointwise operations for interval functions are defined in the usual way:

$$(f + g)(x) = f(x) + g(x),\ (\alpha \cdot f)(x) = \alpha \cdot f(x),\ (f \times g)(x) = f(x) \times g(x). \quad (1)$$

It is easy to see that the set of S-continuous functions is closed under the above pointwise operations while the set of H-continuous functions is not, see [2], [4]. Hence the significance of the following theorem.

Theorem 4. *For any $f, g \in \mathbb{H}(\Omega)$ and $\alpha \in \mathbb{R}$ the functions $f+g$, $\alpha \cdot f$ and $f \times g$ are quasi-minimal S-continuous functions.*

Proof. Denote by D_{fg} the subset of Ω where both f and g assume point values. Then $f+g$ assumes point values on D_{fg}. According to Theorem 2 the set D_{fg} is dense in Ω which in terms of Theorem 3 implies that $f+g$ is quasi-minimal. The quasi-minimality of $\alpha \cdot f$ and $f \times g$ is proved in a similar way.

We define the algebraic operations on $\mathbb{H}(\Omega)$ using Theorem 4. We denote these operations respectively by $\oplus$, $\odot$ and $\otimes$ so that distinction from the pointwise operations can be made.

Definition 1. *Let $f, g \in \mathbb{H}(\Omega)$ and $\alpha \in \mathbb{R}$. Then*

$$f \oplus g = [f+g], \ \alpha \odot f = [\alpha \cdot f], \ f \otimes g = [f \times g]. \tag{2}$$

Theorem 5. *The set $\mathbb{H}(\Omega)$ is a commutative algebra with respect to the operations $\oplus$, $\odot$ and $\otimes$ given in (2).*

The proof will be omitted; it involves standard techniques and is partially discussed in [5].

3 The concept of ultra-arithmetical functoid

Functoid is a structure resulting from the ultra-arithmetical approach to the solution of problems in functional spaces. The aim of ultra-arithmetic is the development of structures, data types and operations corresponding to functions for direct digital implementation. On a digital computer equipped with ultra-arithmetic, problems associated with functions are solvable, just as now we solve algebraic problems [9]. Ultra-arithmetic is developed in analogy with the development of computer arithmetic.

Let $\mathcal{M}$ be a space of functions and let M be a finite dimensional subspace spanned by $\Phi_N = \{\varphi_k\}_{k=0}^N$. Every function $f \in \mathcal{M}$ is approximated by $\tau_N(f) \in M$. The mapping τ_N is called rounding (in analogy with the rounding of numbers) and the space M is called a screen of $\mathcal{M}$. Every rounding must satisfy the requirement (invariance of rounding on the screen): $\tau_N(f) = f$ for every $f \in M$. Every function $f = \sum_{i=0}^N \alpha_i \varphi_i \in M$ can be represented by its coefficient vector $\nu(f) = (\alpha_0, \alpha_1, \ldots, \alpha_N)$. Therefore the approximation of the functions in $\mathcal{M}$ is realized through the mappings $\mathcal{M} \xrightarrow{\tau_N} M \overset{\nu}{\longleftrightarrow} K^{N+1}$, where K is the scalar field of $\mathcal{M}$ (i.e. $K = \mathbb{R}$ or $K = \mathbb{C}$). Since ν is a bijection we can identify M and K^{N+1} and consider only the rounding τ_N.

In $\mathcal{M}$ we consider the operations addition (+), multiplication by scalars (.), multiplication of functions ($\times$) and integration ($\int$) defined in the conventional way. By the semimorphism principle τ_N induces corresponding operations in M:

$$f \boxed{\circ} g = \tau_N(f \circ g)\ , \quad \circ \in \{+, ., \times\};$$
$$\boxed{\int} f = \tau_N\left(\int f\right).$$

The structure $(M, \boxplus, \boxdot, \boxtimes, \boxed{\int})$ is called an (ultra-arithmetical) functoid [10].

4 A functoid in $\mathbb{H}(\Omega)$

To simplify the matters we consider the function space of all bounded H-continuous functions on $\Omega = (-1,1)$. Furthermore, since we shall often use a shift of the argument, we assume that all functions are produced periodically (period 2) over $\mathbb{R}$. Hence we denote the space under consideration by $\mathbb{H}_{per}(-1,1)$. All algebraic operations on $\mathbb{H}_{per}(-1,1)$ are considered in terms of Definition 1. For simplicity we denote them as the operations for reals. Namely, addition is "+" and a space is interpreted as multiplication, where the context shows whether this is a multiplication by scalars or product of functions. In particular, note that indicating the argument of a function in a formula does not mean pointwise operation. Denote by s_1 the H-continuous function given by

$$s_1(x) = \begin{cases} x, & \text{if } x \in (-1,1), \\ [-1,1], & \text{if } x = \pm 1; \end{cases}$$

and produced periodically over the real line. Since the integrals of $\underline{s}_1$ and $\overline{s}_1$ are equal over any interval the integral of s_1 is a usual real function. We construct iteratively the sequence of periodic splines $s_1, s_2, s_3, \ldots$ using

$$s_{j+1} = \int s_j(x)dx + c,$$
$$\int_{-1}^{1} s_{j+1}(x)dx = s_{j+2}(1) - s_{j+2}(-1) = 0.$$

Theorem 6. *Let $f \in \mathbb{H}_{per}(-1,1)$ be given. Assume that there exists a finite set $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \subset (-1,1]$ such that f assumes real values and is p times differentiable on $(-1,1] \setminus \Lambda$ with the p-th derivative in $L^2(-1,1)$. Then f has a unique representation in the form*

$$f(x) = a_0 + \sum_{l=1}^{m}\sum_{j=1}^{p} a_{jl} s_j(x+1-\lambda_l) + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} b_k e^{ik\pi x}, \tag{3}$$

where $\sum\limits_{\substack{k=-\infty \\ k\neq 0}}^{\infty} b_k e^{ik\pi x}$ is p times differentiable with its p-th derivative in $L^2(-1,1)$. Furthermore, the coefficients are given by:

$$a_0 = \frac{1}{2}\int_{-1}^{1} f(x)dx,$$
$$a_{jl} = \frac{1}{2}\left(\frac{d^{j-1}f}{dx^{j-1}}(\lambda_l - 0) - \frac{d^{j-1}f}{dx^{j-1}}(\lambda_l + 0)\right), \quad \begin{matrix} j = 1, \ldots, p \\ l = 1, \ldots, m \end{matrix}$$
$$b_k = \frac{1}{2(ik\pi)^p}\int_{-1}^{1} \frac{d^p f(x)}{dx^p} e^{-ik\pi x} dx\ ,\ k = \pm 1, \pm 2, \ldots$$

The proof uses standard techniques and will be omitted.

Function f is approximated by

$$\rho_{Np}(f;x) = a_0 + \sum_{l=1}^{m}\sum_{j=1}^{p} a_{jl} s_j(x+1-\lambda_l) + \sum_{\substack{k=-N \\ k\neq 0}}^{N} b_k e^{ik\pi x} \tag{4}$$

with a rounding error

$$\begin{aligned} |f(x) - \rho_{Np}(f;x)| \; &= \; \left| \sum_{|k|>N} b_k e^{ik\pi x} \right| \; \leq \; \sum_{|k|>N} |b_k| \\ \leq & \left(\sum_{|k|>N} (k\pi)^{2p} |b_k|^2 \right)^{\frac{1}{2}} \left(\sum_{|k|>N} \frac{1}{(k\pi)^{2p}} \right)^{\frac{1}{2}} \qquad (5) \\ \leq & \left(\frac{1}{2}\int_{-1}^{1} \left(\frac{d^p f(x)}{dx^p} \right)^2 dx - \left(\sum_{l=1}^{m} a_{pl} \right)^2 - \sum_{\substack{k=-N \\ k\neq 0}}^{N} (k\pi)^{2p} |b_k|^2 \right)^{\frac{1}{2}} \left(\frac{2}{(2p-1)\pi^{2p} N^{2p-1}} \right)^{\frac{1}{2}} \\ = \; & o\left(\frac{1}{N^{p-\frac{1}{2}}} \right) \end{aligned}$$

Motivated by the above we consider a screen in $\mathbb{H}_{per}(-1,1)$ comprising the subspace M spanned by the basis

$$\{s_0(x)\} \cup \{s_j(x+1-\lambda_l) : j = 0,1,..,p,\ l = 1,..m\} \cup \{e^{ik\pi x} : k = 0, \pm 1, ..., \pm N\},$$

where $p, m, N \in \mathbb{N}$ and $\{\lambda_1, \lambda_2, ..., \lambda_m\} \subset (-1,1]$ are parameters with arbitrary but fixed values. Here s_0 is the function which is constant 1 on $\mathbb{R}$. Defining a rounding from $\mathbb{H}_{per}(-1,1)$ to M is still an open problem. However, for functions of the type described in Theorem 6 the rounding is defined through ρ_{Np}. Furthermore, to define a functoid we only need to know how to round the functions resulting from operations in M. For this purpose the rounding ρ_{Np} is sufficient. Naturally, since M is a subspace it is closed under the operations addition and multiplication by scalars. Furthermore, to define multiplication of functions and integration we only need to define these operations on the elements of the basis. The products of the functions in the basis are given by

$$\begin{aligned} & s_{q_1}(x+1-\lambda_{l_1}) s_{q_2}(x+1-\lambda_{l_2}) \\ & \quad = \sum_{j=q_1}^{q_1+q_2} \binom{j-1}{q_1-1} s_{q_1+q_2-j}(1+\lambda_{l_1}-\lambda_{l_2}) s_j(x+1-\lambda_{l_1}) \qquad (6) \\ & \qquad + \sum_{j=q_2}^{q_1+q_2} \binom{j-1}{q_2-1} s_{q_1+q_2-j}(1-\lambda_{l_1}+\lambda_{l_2}) s_j(x+1-\lambda_{l_2}), \\ & e^{ik_1\pi x} e^{ik_2\pi x} = e^{i(k_1+k_2)\pi x}, \qquad (7) \end{aligned}$$

$$s_q(x+1-\lambda_l)e^{in\pi x}$$
$$= \sum_{j=q}^{p}(-1)^n e^{i(\lambda_l-1)\pi}\binom{j-1}{q-1}(in\pi)^{j-q}s_j(x+1-\lambda_l) + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\beta_k e^{ik\pi x}, \quad (8)$$

where in (8) the coefficients β_k are given by

$$\beta_k = \frac{(-1)^{k-n-1}n^{p-q}}{k^p(i\pi)^q}\sum_{r=0}^{q-1}\binom{p}{r}\left(\frac{n}{k-n}\right)^{q-r}, \quad \text{if } k \neq 0, n,$$
$$\beta_n = \binom{p}{q}(in\pi)^{-q}.$$

For the respective integrals we have

$$\int s_j(x)dx = s_{j+1}(x) \ , \quad j = 1, ..., p, \qquad (9)$$
$$\int e^{ik\pi x}dx = \frac{1}{ik\pi}e^{ik\pi x} \ , \quad k = 0, \pm 1, ..., \pm N. \qquad (10)$$

Obviously in formulas (6)–(9) we obtain splines s_j with $j > p$ and exponents $e^{ik\pi x}$ with $|k| > N$ which need to be rounded. Using that

$$s_j(x+1-\lambda_l) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\frac{(-1)^{k-1}e^{i(1-\lambda_l)\pi}}{(ik\pi)^j}e^{ik\pi}$$

all that needs to be rounded in Fourier series which is done by truncation. Note that at any time we truncate a Fourier series of a function which is at least p times differentiable with its p-th derivative in $L^2(-1,1)$. Hence the uniform norm of the error is $o\left(\frac{1}{N^{p-\frac{1}{2}}}\right)$. The integration of s_0, when it arises in practical problems, should be handled with special care since $\int s_0(x)dx = s_1(x)$ holds only on $(-1,1)$.

5 Application to the wave equation

We consider the wave equation in the form

$$u_{tt}(x,t) - u_{xx}(x,t) = \rho(t)u(x,t) + \phi(x,t)$$
$$u(x,0) = g_1(x), \ u_t(x,0) = g_2(x)$$

with periodic boundary conditions at $x = -1$ and $x = 1$, assuming that g_1, g_2, ϕ or some of their space derivatives may be discontinuous but the functions can be represented as a spline-Fourier series (3) of the space variable. An approximation to the solution is sought in the form

$$u(x,t) \ = \ a_0(t) + \sum_{l=1}^{m}\sum_{j=1}^{p}\sum_{\delta=-1}^{1} a_{lj\delta}(t)s_j(x+\delta t+1-\alpha_l)$$

$$+\sum_{\substack{k=-N\\k\neq 0}}^{N} b_k(t)e^{ik\pi x} \tag{11}$$

where $\alpha_l,\ l=1,\dots,m$, are points in $(-1,1]$ where the data functions or some of their first $p-1$ derivatives may be discontinuous.

The following Newton-type iterative procedure is applied

$$u^{(r+1)}=(1-\lambda)u^{(r)}+\lambda\left(g+\frac{1}{2}\iint\limits_{G(x,t)}\rho u^{(r)}\right),$$

where $G(x,t)$ is the triangle with vertices (x,t), $(x-t,0)$, $(x+t,0)$ and

$$g(x,t)=\frac{1}{2}\left(g_1(x+t)+g_1(x-t)+\int_{x-t}^{x+t}g_2(\theta)d\theta+\iint\limits_{G(x,t)}\phi(y,\theta)dyd\theta\right).$$

The essential part of each iteration is the evaluation of the integral. This can be done successfully using the arithmetic in the functoid discussed in the preceding section. We also have to choose some form of representation of the coefficients $a_{mj}(t)$, $b_k(t)$. Here we carry out the computations representing those coefficients as polynomials of t. The following formulas are used:

$$\iint\limits_{G(x,t)}\frac{\theta^q}{q!}s_j(y)\mathrm{d}y\mathrm{d}\theta=s_{j+q+2}(x+t)+(-1)^q s_{j+q+2}(x-t)-2\sum_{\substack{l=0\\l-\text{even}}}^{q}\frac{t^{q-l}}{(q-l)!}s_{j+l+2}(x)$$

$$\iint\limits_{G(x,t)}\frac{\theta^q}{q!}s_j(y+\theta)\mathrm{d}y\mathrm{d}\theta=\sum_{l=0}^{q+1}\left(-\frac{1}{2}\right)^l\frac{t^{q+1-l}}{(q+1-l)!}s_{j+l+1}(x+t)-\left(-\frac{1}{2}\right)^{q+1}s_{j+q+2}(x-t)$$

$$\iint\limits_{G(x,t)}\frac{\theta^q}{q!}s_j(y-\theta)\mathrm{d}y\mathrm{d}\theta=\left(\frac{1}{2}\right)^{q+1}s_{j+q+2}(x+t)-\sum_{l=0}^{q+1}\left(\frac{1}{2}\right)^l\frac{t^{q+1-l}}{(q+1-l)!}s_{j+l+1}(x-t)$$

$$\iint\limits_{G(x,t)}\frac{\theta^q}{q!}e^{ik\pi y}\mathrm{d}y\mathrm{d}\theta=\frac{1}{(ik\pi)^{q+2}}\left(e^{ik\pi(x+t)}+(-1)^q e^{ik\pi(x-t)}-2\sum_{\substack{l=0\\l-\text{even}}}^{q}\frac{t^{q-l}}{(ik\pi)^{q-l}(q-l)!}e^{ik\pi x}\right)$$

$$=2\sum_{\substack{l=0\\l-\text{even}}}^{\infty}(ik\pi)^l\frac{t^{q+l+2}}{(q+l+2)!}e^{ik\pi x}.$$

The splines s_j for $j>p$ as well as the infinite series in the last formula above are approximated by a partial sum of the respective Fourier series using the rounding ρ_{Np}. As shown in Section 4 the truncation error is $o\left(\frac{1}{N^{p-\frac{1}{2}}}\right)$. The main advantage of the method is that it produces highly accurate results for relatively small values of p and N for non-smooth data functions. Numerical experiments using $p = 5$ and $N = 5$ produced 4–5 correct decimal digits of the solution.

6 Conclusion

H-continuous functions appear in many fields of mathematics, notably in the analysis of nonlinear PDE's. Hence the need of a methodology for numerical computations with H-continuous functions. We propose a method based on the fact that H-continuous functions form a linear space when addition is defined in a suitable way. Our method makes use of the ultra-arithmetic approach for the construction of a relevant functoid. The method has been tested numerically for the solution of the wave equation for non-smooth boundary conditions. Highly accurate results have been achieved for rather small number of base functions, i. e. small dimensions of the underlying linear space.

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