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ON THE RELATION INCLUSION OF ZONOTOPES IN THE
PLANE*

Svetoslav M. Markov, Antonio C. da Rocha Costa, Gracaliz P. Dimuro

The theory of quasivector spaces has been applied for a class of zonotopes in the plane defined as positive combinations of basic centered segments. This leads to an implicit presentation of the zonotopes by means of two vectors: one for the center and one for the centered zonotope obtained by translation of the original zonotope in the origin. Using this presentation, the order relation inclusion of zonotopes in the plane has been studied. More specifically, sufficient conditions for inclusion have been stated in terms of the implicit presentation.

1. Introduction. Certain practically important systems, such as various classes of convex bodies, zonotopes, intervals, interval vectors and functions, stochastic numbers, etc., are abelian cancellative monoids with respect to addition (in Minkowski sense). With respect to multiplication by scalars they satisfy the axioms of a linear space with one exception: the second distributive law is weakened up to a so-called quasidistributive law, stating that distributivity must hold only for equally signed scalars. These spaces naturally involve a partial order relation — inclusion, which is isotone with respect to both addition and multiplication by scalar. Every such space can be embedded into an abelian additive group. The latter, when equipped with multiplication by scalars, turns into a so-called quasivector space.

Every quasivector space is a direct sum of a vector space and a quasivector space of symmetric elements called symmetric quasivector space [2], [3]. In the finite case every vector space is isomorphic to $(\mathbb{R}^n, +, \mathbb{R}, \cdot)$ and every symmetric quasivector space is isomorphic to a similar canonic space $(\mathbb{R}^k, +, \mathbb{R}, *)$ which differs from a vector space by its multiplication by scalars “*”. In practice this means that intervals and zonotopes are naturally presented by their centers and symmetrical (origin centered) parts. Using such presentation, we discuss the computation with zonotopes in the plane, and more specifically, the inclusion relation between zonotopes in the plane. Our purpose is to use only the implicit presentation of zonotopes as Minkowski sums of linear segments [1], [4]. Within such frames using ideas from the theory of quasivector spaces, we formulate sufficient conditions for inclusion (containment) of a class of zonotopes in the plane. In Section 2 we briefly introduce some notation and give some properties of quasivector

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spaces. Section 3 is devoted to the presentation of zonotopes, and Section 4 – to the inclusion of zonotopes in the plane.

2. Quasivector Spaces. By \mathbb{R} we denote the set of reals; we use the same notation for the linearly ordered field of reals $\mathbb{R} = (\mathbb{R}, +, \cdot, \leq)$. For any integer $n \geq 1$ denote by \mathbb{R}^n the set of all n -tuples $(\alpha_1, \alpha_2, \dots, \alpha_n)$, where $\alpha_i \in \mathbb{R}$. The set \mathbb{R}^n forms a vector lattice $\mathbb{V}^n = (\mathbb{R}^n, +, \mathbb{R}, \cdot, \leq)$ under addition, multiplication by scalars and the partial order “ \leq ”. We recall that $(\alpha_1, \alpha_2, \dots, \alpha_n) \leq (\beta_1, \beta_2, \dots, \beta_n)$ means $\alpha_i \leq \beta_i$ for all $i = 1, \dots, n$.

Quasivector spaces are defined as follows [2], [3]: A *quasivector space* (over \mathbb{R}), denoted $(\mathcal{Q}, +, \mathbb{R}, *)$, is an abelian group $(\mathcal{Q}, +)$ with multiplication by scalars “ $*$ ”: $\mathbb{R} \times \mathcal{Q} \rightarrow \mathcal{Q}$, such that for $a, b, c \in \mathcal{Q}$, $\alpha, \beta, \gamma \in \mathbb{R}$: $\gamma*(a+b) = \gamma*a + \gamma*b$, $\alpha*(\beta*c) = (\alpha\beta)*c$, $1*a = a$, $(\alpha+\beta)*c = \alpha*c + \beta*c$, if $\alpha\beta \geq 0$. The last property is called quasidistributive law.

Let a be an element of a quasivector space $(\mathcal{Q}, +, \mathbb{R}, *)$, $a \in \mathcal{Q}$. The operator $\neg a = (-1)*a$ is called *negation*. We write $a \neg b = a + (-b)$. Due to the quasidistributive law $a \neg a = 0$ may not hold, thus generally negation is different from opposite. An element $a \in \mathcal{Q}$ with $a \neg a = 0$ is called *linear*. An element $a \in \mathcal{Q}$ with the property $\neg a = a$ is called *centered* or (*origin*) *symmetric*.

The canonic symmetric quasivector space. Consider the set \mathbb{R}^k , $k \geq 2$, of all k -tuples $(\alpha_1, \alpha_2, \dots, \alpha_k)$, $\alpha_i \in \mathbb{R}$, with the following operations:

$$\begin{aligned} (\alpha_1, \alpha_2, \dots, \alpha_k) + (\beta_1, \beta_2, \dots, \beta_k) &= (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_k + \beta_k), \\ \gamma * (\alpha_1, \alpha_2, \dots, \alpha_k) &= (|\gamma|\alpha_1, |\gamma|\alpha_2, \dots, |\gamma|\alpha_k), \gamma \in \mathbb{R}. \end{aligned}$$

It is easy to check that the space $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$ thus defined is a quasivector space over \mathbb{R} . Negation $\neg(\alpha_1, \alpha_2, \dots, \alpha_k) = (\alpha_1, \alpha_2, \dots, \alpha_k)$ is the same as identity, which means that all elements of $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$ are symmetric. Due to this the space $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$ is called symmetric quasivector space. The opposite operator in \mathbb{S}^k is: $\text{opp}(\alpha_1, \alpha_2, \dots, \alpha_k) = (-\alpha_1, -\alpha_2, \dots, -\alpha_k)$.

If we define in $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$ a multiplication by scalars “ \cdot ” by means of:

$$\gamma \cdot \alpha = \begin{cases} \gamma * \alpha, & \text{if } \gamma \geq 0, \\ \gamma * \text{opp}(\alpha), & \text{if } \gamma < 0, \end{cases}$$

for $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$, $\gamma \in \mathbb{R}$, then we obtain the familiar vector (linear) space $\mathbb{V}^k = (\mathbb{R}^k, +, \mathbb{R}, \cdot)$. Thus, we see that a symmetric quasivector space differs from a linear space only by the definition of multiplication by scalar.

Consider the direct sum $\mathbb{V}^l \oplus \mathbb{S}^k$ of an l -dimensional vector space $\mathbb{V}^l = (\mathbb{R}^l, +, \mathbb{R}, \cdot)$ and a k -dimensional symmetric quasivector space $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$. The elements of $\mathbb{V}^l \oplus \mathbb{S}^k$ are denoted as $a = (a'; a'')$ with $(a'; 0) \in \mathbb{V}^l$, $(0; a'') \in \mathbb{S}^k$. For $a, b \in \mathbb{V}^l \oplus \mathbb{S}^k$, $\gamma \in \mathbb{R}$, we define:

$$\begin{aligned} (1) \quad a + b &= (a'; a'') + (b'; b'') = (a' + b'; a'' + b''), \\ (2) \quad \gamma * a &= \gamma * (a'; a'') = (\gamma \cdot a'; \gamma * a'') = (\gamma a'; |\gamma| a''). \end{aligned}$$

It is immediately checked that $\mathbb{V}^l \oplus \mathbb{S}^k$ is a quasivector space (in general, neither linear, nor symmetric). Negation in $\mathbb{V}^l \oplus \mathbb{S}^k$ is $\neg(a'; a'') = (-a'; a'')$ and opposite is: $\text{opp}(a'; a'') = (-a'; -a'')$. The composition of negation and opposite is a new automorphic transformation (involution) called *conjugation* (*dual operator*): $(a'; a'')_- = \neg \text{opp}(a'; a'') = \neg(-a'; -a'') = (a'; -a'')$.

We use the notation $a_- = \neg\text{opp}(a) = \text{opp}(\neg a)$ for the dual operator in a general quasivector space. The relations $\neg\text{opp}(a) = \text{opp}(\neg a) = a_-$ imply $\text{opp}(a) = \neg(a_-) = (\neg a)_-$, shortly $\text{opp}(a) = \neg a_-$. Thus, the symbolic notation $\neg a_-$ can be used instead of $\text{opp}(a)$, and, for $a \in \mathcal{Q}$ we can write $a \neg a_- = 0$, resp. $\neg a_- + a = 0$.

Assume that \mathcal{Q} is a quasivector space. The subsets of linear and centered elements $\mathcal{Q}' = \{a \in \mathcal{Q} \mid a \neg a = 0\}$, resp. $\mathcal{Q}'' = \{a \in \mathcal{Q} \mid a = \neg a\}$ form subspaces of \mathcal{Q} . The subspace \mathcal{Q}' is a vector space. The space $\mathcal{Q}' = \{a \in \mathcal{Q} \mid a \neg a = 0\}$ is called the *linear subspace* of \mathcal{Q} and the space $\mathcal{Q}'' = \{a \in \mathcal{Q} \mid a = \neg a\}$ is called the *symmetric subspace* or *centered subspace* of \mathcal{Q} .

Theorem 1. [2] *For every quasivector space \mathcal{Q} we have $\mathcal{Q} = \mathcal{Q}' \oplus \mathcal{Q}''$. More specifically, for every $x \in \mathcal{Q}$ we have $x = u + v$ with unique $u = (1/2) * (x + x_-) \in \mathcal{Q}'$, and $v = (1/2) * (x \neg x) \in \mathcal{Q}''$.*

Theorem 1 implies that computation in a quasivector space $\mathcal{Q} = \mathcal{Q}' \oplus \mathcal{Q}''$ is reduced by means of (1), (2) to computation in the spaces \mathcal{Q}' and \mathcal{Q}'' . All vector space concepts, such as subspace, sum and direct sum, linear combination, basis, etc., are extended to symmetric quasivector spaces [2]. As we know, if the vector space \mathcal{Q}' is finite, spanned over n basic vectors, then it is isomorphic to \mathbb{V}^n . A similar result holds true for a symmetric quasivector space as follows.

Theorem 2. [2] *Any symmetric quasivector space over \mathbb{R} , with a basis of k elements, is isomorphic to $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$.*

3. Computation with Zonotopes in the Plane. Centered zonotopes. A centrally symmetric convex body with center at the origin is called *centered convex body* (cf. [5], p. 383).

In what follows we restrict ourselves to 2D-zonotopes, that is zonotopes in the Euclidean plane \mathbb{E}^2 with a fixed coordinate system Oxy . Zonotopes are special convex bodies and have several different presentations. In this work we shall make use of the presentation based on the Minkowski sum of segments. The latter presentation is in accordance with the general theory of quasivector spaces.

Every unit vector $e = (\cos \varphi, \sin \varphi) \in \mathbb{E}^2$, $\varphi \in [0, \pi)$, defines a centered segment \tilde{e} with endpoints $-e$ and e : $\tilde{e} = \text{conv}\{-e, e\} = \{\lambda e \mid \lambda \in [-1, 1]\}$, where “conv” means the convex hull, see [5]. In the sequel Ov denotes the line passing through the origin and the point v and \tilde{v} (or \tilde{v}) denotes the centered (origin symmetric) segment on the line Ov comprising the points between $-v$ and v , that is $\tilde{v} = \text{conv}\{-v, v\}$. Note that v is a vector, $v \in \mathbb{R}^2$, whereas \tilde{v} is a centered linear segment. The latter is the simplest example (together with a point) of a zonotope. The sum of two (centered) segments $\alpha_i * \tilde{e}^{(i)} + \alpha_j * \tilde{e}^{(j)}$ is a (centered) parallelogram in the plane.

For $\rho \in \mathbb{R}$ denote $s = \rho e$. Multiplication of a unit centered segment \tilde{e} by a scalar $\rho \in \mathbb{R}$ is:

$$\tilde{s} = \rho * \tilde{e} = (\rho e)\tilde{ } = \text{conv}\{-s, s\} = \{\lambda \rho e \mid \lambda \in [-1, 1]\}.$$

Multiplication of a centered (not necessarily unit) segment $(\rho e)\tilde{ }$ by a scalar $\gamma \in \mathbb{R}$ satisfies $\gamma * (\rho e)\tilde{ } = ((\gamma \rho) e)\tilde{ } = (\gamma \rho) * \tilde{e}$. Note that $-1 * \tilde{s} = \tilde{s}$; more generally, $-\rho * \tilde{s} = \rho * \tilde{s}$ (for comparison, $\rho s \neq -\rho s$). Thus the “quasivector” multiplication by scalars “*” is different from the linear multiplication by scalars “*”, both operations coinciding for nonnegative scalars.

Assume that we are given a mesh of k numbers (angles) φ_i in the interval $[0, \pi)$, such that

$$(3) \quad 0 \leq \varphi_1 < \varphi_2 < \dots < \varphi_k < \pi.$$

A system of the form (3) is called *regular*. Every φ_i defines a unit vector $e^{(i)} = (\cos \varphi_i, \sin \varphi_i)$, respectively a centered unit segment: $\tilde{e}^{(i)} = \text{conv}\{-e^{(i)}, e^{(i)}\}$. The induced systems of unit vectors, resp. segments:

$$(4) \quad e^{(1)}, e^{(2)}, \dots, e^{(k)}, \quad \tilde{e}^{(1)}, \tilde{e}^{(2)}, \dots, \tilde{e}^{(k)},$$

are also called *regular*. The elements of the systems (4) are cyclically anticlockwise ordered; the point $e^{(1)}$ lies on the Ox axis of the plane coordinate system Oxy .

In the sequel we shall assume that k is an integer ≥ 2 . For $\alpha_i \geq 0$, $i = 1, \dots, k$, the vectors $\alpha_i e^{(i)} = (\alpha_i \cos \varphi_i, \alpha_i \sin \varphi_i)$ induce centered segments $\alpha_i * \tilde{e}^{(i)} = (\alpha_i e^{(i)})^\circ$. The *positive combination* of unit centered segments

$$(5) \quad c = \sum_{i=1}^k \alpha_i * \tilde{e}^{(i)}, \quad \alpha_i \geq 0,$$

is a centered *zonotope*. By adding a translate vector v we obtain a translated zonotope $z = v + c$.

The sum of the segments $s_i = \alpha_i * \tilde{e}^{(i)}$ is understood in Minkowski sense (vector sum):

$$(6) \quad \sum_{i=1}^k s_i = \left\{ \sum_{i=1}^k \gamma_i e^{(i)} \mid \gamma_i \in [-\alpha_i, \alpha_i] \right\}.$$

The zonotope (5) has $2k$ vertices: $v^{(1)}, v^{(2)}, \dots, v^{(k)}, -v^{(1)}, -v^{(2)}, \dots, -v^{(k)}$ [5]; symbolically we have $z = \text{conv}\{v^{(1)}, v^{(2)}, \dots, v^{(k)}, -v^{(1)}, -v^{(2)}, \dots, -v^{(k)}\}$, where

$$(7) \quad \begin{aligned} v^{(1)} &= \alpha_1 e^{(1)} + \alpha_2 e^{(2)} + \dots + \alpha_{k-1} e^{(k-1)} + \alpha_k e^{(k)}, \\ v^{(2)} &= -\alpha_1 e^{(1)} + \alpha_2 e^{(2)} + \dots + \alpha_{k-1} e^{(k-1)} + \alpha_k e^{(k)}, \\ &\dots \\ v^{(i)} &= -\alpha_1 e^{(1)} - \dots - \alpha_{i-1} e^{(i-1)} + \alpha_i e^{(i)} + \dots + \alpha_k e^{(k)}, \\ &\dots \\ v^{(k)} &= -\alpha_1 e^{(1)} - \alpha_2 e^{(2)} + \dots - \alpha_{k-1} e^{(k-1)} + \alpha_k e^{(k)}. \end{aligned}$$

The vertices $v^{(1)}, v^{(2)}, \dots, v^{(k)}$ given by (7) are lying in cyclic order anticlockwise in a half-plane between the vectors $v^{(1)}$ and $v^{(k)} = -v^{(1)} + 2\alpha_k e^{(k)}$.

In accordance with Theorem 1 zonotopes in $\mathbb{V}^2 \oplus \mathbb{S}^k$ will be presented as $z = u + v = (z'; 0) + (0, z'')$, where z' is the translate vector (center) in the plane \mathbb{V}^2 and z'' is the symmetric zonotope from \mathbb{S}^k . Thus every zonotope $z \in \mathbb{V}^2 \oplus \mathbb{S}^k$ can be presented in the form:

$$(8) \quad z = \sum_{i=1}^k z_k' e^{(i)} + \sum_{i=1}^k z_k'' * \tilde{e}^{(i)},$$

that is $z = (z'; z'')$, where the k -tuple $z' = (z_1', \dots, z_k') \in \mathbb{R}^k$ presents the translating vector $z' = \sum_{i=1}^k z_k' e^{(i)}$ and $z'' = (z_1'', \dots, z_k'') \in \mathbb{R}^k$ presents the symmetric zonotope $z'' = \sum_{i=1}^k z_k'' * \tilde{e}^{(i)}$. The form $z = (z'; z'') = (z_1', \dots, z_k'; z_1'', \dots, z_k'')$ will be further called *M-presentation* or *M-decomposition* to remind that this form is used in the Minkowski

sum (6).

Relation (8) can also be written in the form $z = \sum_{i=1}^k z_i$, where

$$(9) \quad z_i = (z_i'; z_i'') = z_k' e^{(i)} + z_k'' * \tilde{e}^{(i)}, \quad i = 1, \dots, k,$$

are segments on the line $Oe^{(i)}$ centered at $z_k' e^{(i)}$ and having a centered part $z_k'' * \tilde{e}^{(i)}$.

Given a k -tuple $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{R}^k$ the expression $v = \sum_{i=1}^k \gamma_i e^{(i)}$ presents a vector (point) in \mathbb{E}^2 .

For simplicity w. l. o. g. we assume below that the mesh (3) is *uniform*, that is $\varphi_i = \pi(i-1)/k$, $i = 1, \dots, k$; in this case the respective systems (4) are also called *uniform*.

4. Sufficient Conditions for Inclusion of Zonotopes. Consider a class of centered zonotopes in the plane \mathbb{E}^2 with coordinate system Oxy with a basis of k uniform regular unit segments. Define the more general class of all zonotopes in the plane that are translates of centered zonotopes by means of vectors (points) of the plane \mathbb{E} . Every zonotope z in the plane is a Minkowski sum $z = v + c$ of a vector v and a centered zonotope c and hence can be presented implicitly by two k -tuples: one for the translate vector v using its M -presentation, and one for the symmetric part c , which is an element of $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$. The following propositions are straightforward.

Proposition 1. *For two centered zonotopes $c = (0; c''), s = (0; s'')$ we have $c'' \leq s'' \implies cs$.*

Proposition 2. *In the case of n -dimensional intervals $y = (y'; y''), z = (z'; z'')$ the inclusion order $y \subseteq z$ is equivalent to $|z' - y'| \leq z'' - y''$, that is*

$$(10) \quad |z' - y'| \leq z'' - y'' \iff yz.$$

Considering the situation in the plane \mathbb{E}^2 we shall make use of Proposition for $n = 1$ and $n = 2$. We next look for generalizing (10) for zonotopes in the plane.

Note that the points (7) are vertices of the zonotope (5) and hence are included (contained) in the zonotope z , $v^{(i)}z$, $i = 1, \dots, k$.

Lemma 1. *For every $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$, $\alpha \geq 0$, the point $p = \sum_{i=1}^k \pm \alpha_i e^{(i)}$ is included (contained) in the zonotope z defined by (5), pz .*

Proof. All points of the form $\sum_{i=1}^k \pm \alpha_i e^{(i)}$, which are not vertices of z , are interior points of z . This follows from the fact that every point of the form $\pm \alpha_i e^{(i)}$ is an endpoint of the segment $s_i = \alpha_i * \tilde{e}^{(i)}$ and thus participates in the Minkowski sum (6). \square

Using that $\alpha_i * \tilde{e}^{(i)} = |\alpha_i| * \tilde{e}^{(i)}$, we can reformulate Lemma 1 as follows:

Lemma 2. *For a given k -tuple $\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{R}^k$ every point $v = \sum_{i=1}^k \gamma_i e^{(i)}$ is included (contained) in the zonotope $z = \sum_{i=1}^k \gamma_i * \tilde{e}^{(i)}$, symbolically vz .*

Remark. Note that in Lemma 2 we do not require $\gamma \geq 0$.

Our next task is to examine the inclusion of 2D-zonotopes in the space $\mathbb{V}^2 \oplus \mathbb{S}^k$ assuming that $k \geq 2$. To examine the relation ab for two zonotopes $a, b \in \mathbb{V}^2 \oplus \mathbb{S}^k$ we start with the case when a is a point in the plane (translate vector) and b is a centered zonotope. Note that both the points in the plane and the centered zonotopes are special case of zonotopes as elements of $\mathbb{V}^2 \oplus \mathbb{S}^k$.

According to (5), (7) the vector $t^{(1)} = \alpha_1 e^{(1)} + \alpha_2 e^{(2)} + \dots + \alpha_{k-1} e^{(k-1)} + \alpha_k e^{(k)}$, $\alpha_i \geq 0$, is a vertex of the centered zonotope (5): $z = \sum_{i=1}^k \alpha_i * \tilde{e}^{(i)}$. Considered as zonotope itself, the vertex $t^{(1)}$ is contained in (5), that is we have $t^{(1)} \subseteq z$. An M -presentation of $t^{(1)}$ is: $t^{(1)} = (\alpha_1, \dots, \alpha_k; 0, 0, \dots, 0)$. The M -presentation of z in \mathbb{S}^k is $(0, 0, \dots, 0; \alpha_1, \dots, \alpha_k)$. Thus we have

$$(11) \quad t^{(1)} = (\alpha_1, \dots, \alpha_k; 0, 0, \dots, 0) \subseteq (0, 0, \dots, 0; \alpha_1, \dots, \alpha_k) = z.$$

Relation (11) suggests the following formulation of Lemmas 1 and 2.

Proposition 3. *Given a point $v \in \mathbb{V}^2$ with an M -presentation $v = (v'; 0)$, $v' \in \mathbb{R}^k$, and a centered zonotope $c = (0; c'') \in \mathbb{S}^k$, if $|v'| = c''$, then vc .*

Proof. According to Lemma 1 we have $\sum_{i=1}^k \pm c''_i e^{(i)} c$, hence $v = \sum_{i=1}^k v'_i e^{(i)} = \sum_{i=1}^k \pm c''_i e^{(i)} c$. \square

Proposition 4. *Assume that $v \in \mathbb{V}^2$ with M -presentation $v = (v'; 0)$, $v' \in \mathbb{R}^k$, and $s = (0; s'') \in \mathbb{S}^k$. Then $|v'| \leq s'' \implies vs$.*

Proof. According to Proposition a point $v = (v'; 0)$ is included in the centered zonotope $c = (0; c'')$, with $c'' = |v'|$, resp. $c''_i = |v'_i|$, $i = 1, \dots, k$. Hence v is included in every zonotope $s = (0; s'')$ which contains the zonotope c . According to Proposition such is every $s = (0; s'')$, satisfying the condition $s'' \geq c''$, which is equivalent to $s'' \geq |v'|$. \square

Proposition 5. *Assume $v = (v'; 0)$ and $z = (z'; z'')$. If v' , z' are M -presentations, such that $|z' - v'| \leq z''$ then vz .*

Proof. Translate both v and z by the vector $(z'; 0)$. The translated zonotope $c = z - (z'; 0) = (z' - z'; z'') = (0; z'')$ is centered. According to Proposition the translated vector $v - (z'; 0) = (v' - z'; 0)$ belongs to c if $|v' - z'| \leq z''$, which, due to $c'' = z''$, is equivalent to $|z' - v'| \leq z''$. \square

Theorem 3. *Let the zonotopes $a, b \in \mathbb{V}^2 \oplus \mathbb{S}^k$ have M -presentations*

$$a = \sum_{i=1}^k a_i = \sum_{i=1}^k (a'_i; a''_i), \quad b = \sum_{i=1}^k b_i = \sum_{i=1}^k (b'_i; b''_i).$$

If $a_i \subseteq b_i$, for $i = 1, \dots, k$, then $a \subseteq b$.

Proof. Assume $a_i \subseteq b_i$, $i = 1, 2, \dots, k$. From the inclusions $a_i \subseteq b_i$, $i = 1, \dots, k$, and (9), using the inclusion isotonicity of addition, it follows $a = \sum_{i=1}^k a_i \subseteq \sum_{i=1}^k b_i = b$. \square

Concluding remarks. Zonotopes are a suitable tool for bounding regions of uncertainty, enclosing medical images, objects in robotics and technical sciences, etc. To simplify computations, it is desirable to consider zonotopes from a finite-parametric family, with a fixed number of parameters; such a natural family of regular basic vectors has been used in the paper. Our study of the presentation and computation with zonotopes has been guided by the theory of quasivector spaces. As every quasivector space is a direct sum of a linear subspace and a symmetric quasivector subspace, we concentrate on the space of centrally symmetric zonotopes centered at the origin (centered zonotopes) which can be presented as Minkowski sums of centered segments. Our approach is alternative to the approach of support functions, extensively used in the literature on convex bodies. Using such special implicit presentation, we study the inclusion relation between zonotopes in the plane. Within such frames, using ideas from the theory of

quasivector spaces, we formulate several sufficient conditions for inclusion of zonotopes from a certain class.

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Svetoslav Marinov Markov
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Block 8
e-mail: smarkov@bio.bas.bg

Antonio Carlos da Rocha Costa
Gracaliz Pereira Dimuro
Research Group on Mathematics of
Computation Escola de Informatica
Universidade Catolica de Pelotas
Gracaliz Pereira Dimuro
Rua Baro de Santa Tecla 834,
apto 501 96010-140, Pelotas, Brazil
e-mail: rocha@atlas.ucpel.tche.br
e-mail: liz@atlas.ucpel.tche.br

ВЪРХУ РЕЛАЦИЯТА ВКЛЮЧВАНЕ НА ЗОНОТОПИ В РАВНИНАТА

Светослав М. Марков, Антонио Н. Роша Коста, Грасализ П. Димуро

Теорията на квазивекторните пространства се прилага за един клас зонотопи в равнината дефинирани като положителни комбинации от базови центрирани сегменти. Това води до неявно представяне на зонотошите като два вектора – един за центъра и един за центрирания зонотоп получен чрез трансляция на изходния в началото. С помощта на това представяне се изследва релацията включване в равнината. Намерени са достатъчни условия за включване формулирани в термините на неявното представяне.