QUASIVECTOR SPACES AND THEIR RELATION TO VECTOR SPACES

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1 Introduction

Certain practically important systems, such as systems of convex bodies, zonotopes, intervals, interval vectors and functions, stochastic numbers, etc., are (cancellative) quasilinear spaces with respect to addition and multiplication by scalars. These spaces are abelian cancellative monoids with respect to addition. With respect to multiplication by scalars they satisfy the axioms of a linear space with one exception: the second distributive law is weakened up to a so-called "quasidistributive law" stating that distributivity is obligatory only for equally signed scalars.

An important special case of a quasilinear space is the case when the abelian cancellative monoid is an abelian group. A quasilinear space with group structure is very closed to a vector space. Every quasilinear space can be embedded in one with group structure — therefore quasilinear spaces with group structure can be effectively used for computations.

We concentrate on the algebraic properties of quasilinear spaces with group structure. We show that every quasilinear space with group structure is a direct sum of a linear space (of distributive elements) and a space of symmetric elements. The properties of the latter spaces are studied in detail using two different approaches.

The theory of quasilinear spaces of group structure is presented in a comprehensive way and their relation to vector spaces is made clear. Some previous accounts of this theory can be found in [9], [10], [12], [13]. Here we do not consider quasilinear spaces of monoid structure, the interested reader may consult [7], [9] for this case.

2 Quasilinear spaces of monoid structure

Preliminaries. We assume familiarity with elementary set theory and the concepts of set membership $\in$, and the subset relations $\subseteq$ and $\subset$. If X and Y are sets, then the *cartesian product* $X \times Y$ is the set of ordered pairs $\{(x, y) \mid x \in X, y \in Y\}$; if $X = Y$, then $X \times Y$ is written as X^2. Similarly ordered triples, 4-tuples, etc. members of a set X may be constructed.

In what follows we assume that the linearly ordered (l. o.) real field $\mathbb{R} = (\mathbb{R}, +, \cdot, \leq)$ is well known. We next recall and comment the definition of a linear space. In the sequel we use the terms "linear space" and "vector space" as synonyms.

Definition 1. *An algebraic system* $(L, +, \mathbb{R}, \cdot)$ *is a linear space (over the real field* $\mathbb{R}$*), if for* $a, b, c \in L,\ \ \alpha, \beta, \gamma \in \mathbb{R}$*:*

$$(a + b) + c = a + (b + c), \tag{1}$$
$$a + 0 = a, \tag{2}$$
$$a + b = b + a, \tag{3}$$
$$a + (-a) = 0, \tag{4}$$
$$\alpha \cdot (\beta \cdot c) = (\alpha\beta) \cdot c, \tag{5}$$
$$1 \cdot a = a, \tag{6}$$
$$\gamma \cdot (a + b) = \gamma \cdot a + \gamma \cdot b, \tag{7}$$
$$(\alpha + \beta) \cdot c = \alpha \cdot c + \beta \cdot c. \tag{8}$$

Remark. Equality (2) is a brief notation of the property: "for every $a \in L$ there exists a neutral element with respect to addition with the property (2)"; equality (4) means: "for every $a \in L$ there exists an additive inverse (opposite) element with the property (4)". The null element of L, see relation (2), and the null in $\mathbb{R}$, are denoted by the same symbol "0". The additive inverse to $a \in L$ is denoted by "$-a$".

A linear space can be defined also by relaxing the group axiom (4), replacing it by the weaker cancellation law $a + c = b + c \Longrightarrow a = b$. Thus, the following definition of a linear space is equivalent to Definition 1.

Definition 2. *An algebraic system* $(L, +, \mathbb{R}, \cdot)$ *is a linear space (over* $\mathbb{R}$*), if* *for* $a, b, c \in L, \quad \alpha, \beta, \gamma \in \mathbb{R}$*:*

$$\begin{aligned}
(a+b)+c &= a+(b+c), && (9)\\
a+0 &= a, && (10)\\
a+b &= b+a, && (11)\\
a+c=b+c &\Longrightarrow a=b, && (12)\\
\alpha\cdot(\beta\cdot c) &= (\alpha\beta)\cdot c, && (13)\\
1\cdot a &= a, && (14)\\
\gamma\cdot(a+b) &= \gamma\cdot a+\gamma * b, && (15)\\
(\alpha+\beta)\cdot c &= \alpha\cdot c+\beta\cdot c. && (16)
\end{aligned}$$

To check the equivalence between Definition 1 and Definition 2, note that substituting $\alpha = 1, \ \beta = -1$ in (16) gives the relation $0 = c + (-1) \cdot c$, that is the element $(-1) \cdot c$ is the opposite to c, symbolically $-c = (-1) \cdot c$. One can also observe that the condition for existence of a neutral element for addition is redundant, as such is the element $0 \cdot c$. Indeed, using (16) with $\alpha = 1$, $\beta = 0$, we obtain $c + 0 \cdot c = 1 \cdot c + 0 \cdot c = (1 + 0) \cdot c = 1 \cdot c = c$, implying $0 \cdot c = 0$. Definition 2 is methodologically useful for the comparison between linear and quasilinear spaces to be defined next.

A quasilinear space (of monoid structure) is defined by relaxing axiom (16) in the set of linear space axioms (9)–(16). Namely, we have:

Definition 3. *An algebraic system* $(Q, +, \mathbb{R}, *)$ *is a* quasilinear space (of monoid structure, over $\mathbb{R}$*), if for* $A, B, C \in Q, \quad \alpha, \beta, \gamma \in \mathbb{R}$*:*

$$\begin{aligned}
(A+B)+C &= A+(B+C), && (17)\\
A+0 &= A, && (18)\\
A+B &= B+A, && (19)\\
A+C=B+C &\Longrightarrow A=B, && (20)\\
\alpha*(\beta*C) &= (\alpha\beta)*C, && (21)\\
1*A &= A, && (22)\\
\gamma*(A+B) &= \gamma*A+\gamma*B, && (23)\\
(\alpha+\beta)*C &= \alpha*C+\beta*C, \quad \textit{if} \ \ \alpha\beta\ge 0. && (24)
\end{aligned}$$

Relation (24) is called *quasidistributive law.*

Note that the difference between the definitions of a linear and a quasilinear space, namely between Definition 2 given by (9)–(16) and Definition

3 given by (17)–(24), consists just in the restriction $\alpha\beta \geq 0$ in (24). Let us note that this restriction does not permit us to conclude that the quasivector space is a group, as we did in the case with Definition 2. Clearly, every linear space is quasilinear. However, due to the restriction $\alpha\beta \geq 0$ made in (24) a quasilinear space may not be necessarily linear. In fact, all practically important quasilinear spaces, like those of intervals, convex bodies etc., are not linear.

We have used upper case letters in (17)–(24) in order to emphazise that such elements may not have additive inverse (opposite), that is a quasilinear space may not be a group. In particular a quasilinear space may be a group (without beeing linear). Such a quasilinear space is called a *quasivector space (with group structure)*; the precise definition will be given in the next section. We now discuss the defining properties of a quasilinear space in some detail.

Consider first properties related to the operation **addition**. Assume that $\mathcal{Q}$ is a set of elements with a binary operation *addition* "+": $\mathcal{Q} \times \mathcal{Q} \longrightarrow \mathcal{Q}$ and an element null "0", such that for all $A, B, C \in \mathcal{Q}$:

$$(A + B) + C = A + (B + C), \tag{25}$$

$$A + 0 = A, \tag{26}$$

$$A + B = B + A, \tag{27}$$

$$A + C = B + C \implies A = B. \tag{28}$$

Properties (25)–(26) define a semigroup with null, that is a *monoid* $(\mathcal{Q}, +)$. The monoid is abelian (27) and cancellative (28).

An element A of an abelian cancellative (a. c.) monoid $(\mathcal{Q}, +)$ is *invertible*, if there exists $X \in \mathcal{Q}$, such that $A + X = 0$; in this case the (unique) element X is the *opposite* of A, symbolically we write $X = \mathrm{opp}(A)$ (for an opposite element we shall avoid the notation "$-$", of course we shall use the minus notation in the field $\mathbb{R}$).

Denote the set of all invertible elements in $(\mathcal{Q}, +)$ by $\mathcal{Q}_D$. The system $(\mathcal{Q}_D, +)$ is an abelian group, which is not empty due to $0 \in Q_D$.

Consider next properties related to the operation **multiplication by real scalars**. Let $\mathcal{Q}$ be an a. c. monoid and $\mathbb{R}$ be the ordered field of reals. Assume that a multiplication by real scalars $* : \mathbb{R} \times \mathcal{Q} \longrightarrow \mathcal{Q}$ is defined, such that for $A, B, C \in \mathcal{Q}$, $\alpha, \beta, \gamma \in \mathbb{R}$:

$$\alpha * (\beta * C) = (\alpha\beta) * C, \tag{29}$$
$$1 * A = A, \tag{30}$$
$$\gamma * (A + B) = \gamma * A + \gamma * B, \tag{31}$$
$$(\alpha + \beta) * C = \alpha * C + \beta * C, \quad \alpha\beta \geq 0. \tag{32}$$

Note that the quasidistributive law is less restrictive than the *second distributive law* stating that for all elements C of a (linear) system:

$$(\alpha + \beta) * C = \alpha * C + \beta * C, \quad \alpha, \beta \in \mathbb{R}. \tag{33}$$

The elements of a quasilinear space $\mathcal{Q}$ are not required to satisfy (33), however, there may be elements for which (33) holds. Let $C \in \mathcal{Q}$ satisfy (33) for all $\alpha, \beta \in \mathbb{R}$. Substituting $\alpha = 1, \beta = -1$ in (33) gives $0 * C = 1 * C + (-1) * C$, that is $0 = C + (-1) * C$. Hence, the element C is invertible, and opp $C = (-1) * C$. Since the set of all distributive elements forms a linear space, a *distributive element*, that is an element $C \in \mathcal{Q}$, satisfying (33) (for all $\alpha, \beta \in \mathbb{R}$), is also called *linear*.

Definition 3 can be briefly formulated as follows:

Let $(\mathcal{Q}, +)$ *be an a. c. monoid defined by (25)–(28). Assume that multiplication by real scalar "$*$" is defined on* $\mathbb{R} \times \mathcal{Q}$ *satisfying (29)–(32). The algebraic system* $(\mathcal{Q}, +, \mathbb{R}, *)$ *is called a* quasilinear space (of monoid structure, over $\mathbb{R}$*).*

Historical remarks. O. Mayer [16] defines quasilinear spaces as abelian monoids (25)–(27) with multiplication by scalars satisfying (29)–(32) and the additional property

$$0 * A = 0. \tag{34}$$

Thus a quasilinear space in the sense of O. Mayer is an abelian monoid, which is not necessarily cancellative [5], [6], [21], [22]. To summarize, O. Mayer's quasilinear spaces satisfy axioms (25)–(27), (29)–(32), (34). Mayer's axioms can be obtained from the axioms of Definition 3 when replacing the cancellation law (20) by the less restrictive relation (34). Every cancellative quasilinear space is a quasilinear space in the sense of Mayer. For the cancellation law together with (24), gives $0*A+0 = 0*A = (0+0)*A = 0*A+0*A$ implying $0 = 0*A$, showing that relation (34) holds true. O. Mayer's definition does not assume cancellation law so that relation (34) does not follow from (24) and is an independent axiom. Clearly, Mayer's quasilinear spaces are generalizations of quasilinear spaces in the sense of Definition 3. Mayer's spaces are too general for the abstract description of interval and convex body spaces, where cancellation plays an important role. The more restrictive Definition 3 presents a suitable platform for the study of convex body spaces, as this is demonstrated in the sequel.

In what follows we shall be concerned only with *cancellative* quasilinear spaces. For this reason, in the sequel under a *quasilinear space* we shall always mean a *cancellative quasilinear space.*

Multiplication by "-1" is called *negation* in $\mathcal{Q}$ and is denoted by "$\neg$", that is $\neg A = (-1) * A$. Subtraction in $\mathcal{Q}$ is $A + (\neg B) = A + (-1) * B$ and is denoted $A \neg B$.

3 Quasivector Spaces: Definition

Embedding an a. c. monoid in a group. Every abelian monoid $(\mathcal{M}, +)$ with cancellation law induces an abelian group $(\mathcal{D}(\mathcal{M}), +)$, where $\mathcal{D}(\mathcal{M}) = \mathcal{M}^2/\!\sim$ is the *difference (quotient) set* of $\mathcal{M}$ consisting of all pairs (A, B) factorized by the congruence relation $\sim$: $(A, B) \sim (C, D)$ iff $A+D = B+C$, for all $A, B, C, D \in \mathcal{M}$. Addition in $\mathcal{D}(\mathcal{M})$ is defined by $(A, B) + (C, D) = (A+C, B+D)$. The neutral (null) element of $\mathcal{D}(\mathcal{M})$ is the class (Z, Z), $Z \in \mathcal{M}$; due to the existence of null element in $\mathcal{M}$, we have $(Z, Z) \sim (0, 0)$. The opposite element to $(A, B) \in \mathcal{D}(\mathcal{M})$ is $\mathrm{opp}(A, B) = (B, A)$. The mapping $\varphi : \mathcal{M} \longrightarrow \mathcal{D}(\mathcal{M})$ defined for $A \in \mathcal{M}$ by $\varphi(A) = (A, 0) \in \mathcal{D}(\mathcal{M})$ is an *embedding* of monoids. We *embed* $\mathcal{M}$ in $\mathcal{D}(\mathcal{M})$ by identifying $A \in \mathcal{M}$ with the equivalence class $(A, 0) \sim (A + X, X)$, $X \in \mathcal{M}$; all elements of $\mathcal{D}(\mathcal{M})$ admitting the form $(A, 0)$ are called *proper* and the remaining are *improper.* The set of all proper elements of $\mathcal{D}(\mathcal{M})$ is $\varphi(\mathcal{M}) = \{(A, 0) \mid A \in \mathcal{M}\} \cong \mathcal{M}$.

Using the above construction every quasilinear space $(\mathcal{M}, +, \mathbb{R}, *)$ can be embedded into the group $(\mathcal{D}(\mathcal{M}), +)$. Multiplication by scalars "$*$" is extended from $\mathbb{R} \times \mathcal{M}$ to $\mathbb{R} \times \mathcal{D}(\mathcal{M})$ by means of the following natural definition of $* : \mathbb{R} \times \mathcal{D}(\mathcal{M}) \longrightarrow \mathcal{D}(\mathcal{M})$:

$$\gamma * (A, B) = (\gamma * A, \gamma * B), \;\; A, \; B \in \mathcal{M}, \;\; \gamma \in \mathbb{R}. \tag{35}$$

In particular, multiplication by the scalar -1 in $\mathcal{D}(\mathcal{M})$, called *negation*, is

$$\neg(A, B) = (-1) * (A, B) = (\neg A, \neg B), \;\; A, \; B \in \mathcal{M}. \tag{36}$$

Note that (A, B) is proper if and only if $\gamma * (A, B)$ is proper. Indeed, $(A, B) = (C, 0) \Longleftrightarrow A = B + C \Longleftrightarrow \gamma * A = \gamma * (B + C) = \gamma * B + \gamma * C \Longleftrightarrow (\gamma * A, \gamma * B) = (\gamma * C, 0)$.

More about quasilinear spaces with monoid structure can be found in [9]. Here we concentrate on the space $\mathcal{D}(\mathcal{M})$, resp. on quasilinear spaces with group structure, to be briefly called quasivector spaces. In the sequel we shall use lower case roman letters to denote the elements of quasivector spaces of group structure, such as $\mathcal{D}(\mathcal{M})$, writing e. g. $a = (A_1, A_2), A_1, A_2 \in \mathcal{M}$. For example, (36) can be written: $\neg a = (-1) * a$; $a \neg b$ means $a + (\neg b)$, etc. The definition of a **quasivector space** follows.

Definition 4. *A* quasivector space (over the l. o. field $\mathbb{R}$)*, denoted* $(\mathcal{Q}, +, \mathbb{R}, *)$, *is an abelian group* $(\mathcal{Q}, +)$ *with a mapping (multiplication by scalars) "$*$":* $\mathbb{R} \times \mathcal{Q} \longrightarrow \mathcal{Q}$*, such that for* $a, b, c \in \mathcal{Q}, \quad \alpha, \beta, \gamma \in \mathbb{R}$*:*

$$\begin{aligned} \gamma * (a + b) &= \gamma * a + \gamma * b, & (37)\\ \alpha * (\beta * c) &= (\alpha\beta) * c, & (38)\\ 1 * a &= a, & (39)\\ (\alpha + \beta) * c &= \alpha * c + \beta * c, \quad \text{if } \alpha\beta \geq 0. & (40) \end{aligned}$$

Remarks. 1) In (40) and (38) the sum $\alpha+\beta$, resp. the product $\alpha\beta = \alpha\cdot\beta$ and the relation $\alpha\beta \geq 0$ are well-defined in the linearly ordered real field $\mathbb{R} = (\mathbb{R}, +, \cdot, \leq)$. 2) It is easy to see that, if the condition $\alpha\beta \geq 0$ in the *quasidistributive law* (40) is replaced by the condition $\alpha \geq 0$, $\beta \geq 0$, then an equivalent definition is obtained. 3) Note the difference between a quasilinear and a quasivector space in our terminology: a quasivector space is always an additive group, whereas a quasilinear space is an additive a. c. monoid that may not be a group in general.

Clearly, every vector space is a quasivector one and every quasivector space is a quasilinear one. The following proposition is straightforward:

Proposition 1. *Let* $(\mathcal{M}, +, \mathbb{R}, *)$ *be a quasilinear space over* $\mathbb{R}$*, and let* $(\mathcal{Q}, +)$*,* $\mathcal{Q} = \mathcal{D}(\mathcal{M})$*, be the induced abelian group. Let* $* : \mathbb{R} \times \mathcal{Q} \longrightarrow \mathcal{Q}$ *be multiplication by scalars defined by (35). Then* $(\mathcal{Q}, +, \mathbb{R}, *)$ *is a quasivector space over* $\mathbb{R}$.

Conjugate elements. From $\mathrm{opp}(a)+a = 0$ we obtain $\neg\mathrm{opp}(a) \neg a = 0$, that is $\neg\mathrm{opp}(a) = \mathrm{opp}(\neg a)$. The element $\neg\mathrm{opp}(a) = \mathrm{opp}(\neg a)$ is further denoted by a_- and the corresponding operator is called *dualization* or *conjugation*. We say that a_- is the conjugate (or dual) of a.

Relations $\neg\mathrm{opp}(a) = \mathrm{opp}(\neg a) = a_-$ imply $\mathrm{opp}(a) = \neg(a_-) = (\neg a)_-$, which will be shortly denoted $\mathrm{opp}(a) = \neg a_-$. The last notation will be used to denote symbolically the opposite elements instead of the confusing notation $-a$ meaning opposite in algebra and negation in convex and interval analysis. Thus, in a quasivector space we write $a \neg a_- = 0$, resp. $\neg a_- + a = 0$.

Subspaces are defined as in vector spaces. A *subspace of a quasivector space* $(\mathcal{Q}, +, \mathbb{R}, *)$ is a quasivector space $(\mathcal{P}, +, \mathbb{R}, *)$, such that $\mathcal{P} \subseteq \mathcal{Q}$ (the operations in $\mathcal{P}$ are inherited from $\mathcal{Q}$). If $(\mathcal{P}, +, \mathbb{R}, *)$ is a subspace of the quasivector space $(\mathcal{Q}, +, \mathbb{R}, *)$ then, of course, $(\mathcal{P}, +)$ is an abelian subgroup of the abelian group $(\mathcal{Q}, +)$. A sufficient condition for subspace can be formulated as follows. $\mathcal{H}$ is a subspace of the quasivector space $\mathcal{G}$ if and only if $\mathcal{H} \subset \mathcal{G}$ and $\mathcal{H}$ is closed under "+", "$*$", "$_-$", i. e.: i) $a + b \in \mathcal{H}$ for all $a, b \in \mathcal{H}$; ii) $\alpha * c \in \mathcal{H}$ for all $\alpha \in \mathbb{R}$ and $c \in \mathcal{H}$; iii) $a_- \in \mathcal{H}$ for all $a \in \mathcal{H}$.

As an exercise let us show that for any quasilinear space $\mathcal{Q}$ the set of invertable elements forms a subspace of $\mathcal{Q}$. To see this we have to show that the quasilinear space $(\mathcal{Q}_D, +, \mathbb{R}, *)$ is a quasilinear subspace of the quasilinear space $(\mathcal{Q}, +, \mathbb{R}, *)$. To see that $(\mathcal{Q}_D, +, \mathbb{R}, *)$ is closed under "$*$" note that for $A \in \mathcal{Q}_D$ we have $\gamma * (\text{opp}A) = \text{opp}(\gamma * A)$; indeed, $A + (\text{opp}A) = 0$ implies $\gamma * A + \gamma * (\text{opp}A) = 0$. $\Box$

Sum and direct sum of quasivector spaces are defined as in vector spaces. Namely, for two quasivector spaces U, V their sum is $U + V = \{u + v \mid u \in U, v \in V\}$. Let Z be a quasivector space and U, V be subspaces of Z. We say that Z is the *direct sum* of U and V and write $Z = U \oplus V$, if each $z \in Z$ can be uniquely presented in the form $z = u + v$, where $u \in U$, $v \in V$. One can show: 1) a sum $U + V$ is direct, if $u_1 + v_1 = u_2 + v_2$, $u_1, u_2 \in U$, $v_1, v_2 \in V$ imply $u_1 = u_2$, $v_1 = v_2$ (or, equivalently, $u + v = 0$, $u \in U$, $v \in V$ imply $u = 0$, $v = 0$); 2) $Z = U \oplus V \iff Z = U + V$ and $U \cap V = 0$. The elements of $U \oplus V$ are denoted $(u; v)$. Addition in $U \oplus V$ is $(u_1; v_1) + (u_2; v_2) = (u_1 + u_2; v_1 + v_2)$ and multiplication by scalars is $\gamma * (u; v) = (\gamma * u; \gamma * v)$.

4 The Quasidistributive Law

Rules for calculation in quasivector spaces. Let $(\mathcal{Q}, +, \mathbb{R}, *)$ be a quasivector space over $\mathbb{R}$. As $(\mathcal{Q}, +)$ is a group, for every a there exists an opposite element $\text{opp}(a) = \neg a_-$, such that $a \neg a_- = 0$. In a quasivector space the relation $a \neg a = 0$, may not necessarily hold; for due to condition $\alpha\beta \geq 0$ in (40) the equality $(-1) * a + 1 * a = (-1 + 1) * a$ may not be true. This means that $\neg a$ is not generally the opposite of a (unlike in a vector space, where negation and opposite coincide).

Using the group properties, such as $0+a=a$, $\mathrm{opp}(a)+a=0$, $\mathrm{opp}(a+b)=\mathrm{opp}(a)+\mathrm{opp}(b)$, $a+b=a+c \Longrightarrow b=c$, and relations (37)–(40) one can derive rules for calculation in a quasivector space. A list of such rules is summarized in the following

Proposition 2. *Let $(\mathcal{Q},+,\mathbb{R},*)$ be a quasivector space over $\mathbb{R}$. For all $\alpha,\beta,\gamma\in\mathbb{R}$ and for all $a,b,c\in\mathcal{Q}$ the following properties hold: 1) $0*a=0$; 2) $\gamma*0=0$; 3) $\mathrm{opp}(\gamma*a)=\gamma*\mathrm{opp}(a)$; 4) $\neg(\gamma*a)=(-\gamma)*a$; 5) $\gamma*(a\neg b)=\gamma*a\neg\gamma*b$; 6) $\gamma*a=0\Longrightarrow\gamma=0$ or $a=0$; 7) $\gamma*a=\gamma*b\Longrightarrow\gamma=0$ or $a=b$; 8) $(\alpha-\beta)*c=\alpha*c+(-\beta)*c=\alpha*c\neg\beta*c$, $\alpha\beta\le 0$; 9) $(\sum_{i=1}^n\alpha_i)*c=\sum_{i=1}^n\alpha_i*c$, $\alpha_i\ge 0$, $i=1,...,n$; 10) $\alpha*\sum_{i=1}^n c_i=\sum_{i=1}^n\alpha*c_i$.*

Proof. The verification of the above properties is trivial. For example, we prove the first three properties. 1) We have $1*a=(1+0)*a=1*a+0*a$, implying $0*a=0$. 2) If $\gamma=0$ the relation follows from 1); if $\gamma\neq 0$, then $c+\gamma*0=\gamma*((1/\gamma)*c+0)=\gamma*((1/\gamma)*c)=c$, hence $\gamma*0=0$. 3) Assume $\gamma\neq 0$. We have to prove that $\gamma*\mathrm{opp}(a)+\gamma*a=0$, that is, $\mathrm{opp}(a)+(1/\gamma)*(\gamma*a)=0$, which is obviously true. □

Note that $y=a_-$ is the unique solution of the equation: $y\neg a=0$, resp. $\neg y+a=0$. For $\gamma\in\mathbb{R}$ and $a,b\in\mathcal{Q}$ we have the following relations using conjugation: $\gamma*(a\neg b_-)=\gamma*a\neg\gamma*b_-$; $a\neg a_-=\neg a+a_-=0$; $\gamma*a\neg\gamma*a_-=\gamma*(a\neg a_-)=0$; $a+b=0\Longleftrightarrow a=\neg b_-$; $a+\gamma*b=0\Longleftrightarrow a=(-\gamma)*b_-=\neg(\gamma*b_-)$.

In the sequel we shall make use of the binary set $\Lambda=\{+,-\}$ and the function $\sigma:\mathbb{R}\longrightarrow\Lambda$ defined by:

$$\sigma(\gamma)=\begin{cases}+, & \text{if } \gamma\ge 0,\\ -, & \text{if } \gamma<0.\end{cases}$$

The "product" $\lambda\mu$, $\lambda,\mu\in\Lambda$, means $++=--=+$, $+-=-+=-$.

A special symbolic notation. We make the convention $a_+=a$. Then the symbolic notation a_λ for $a\in\mathcal{Q}$, $\lambda\in\Lambda$, makes sense; namely a_λ is either $a=a_+$ or a_- according to the binary value of λ. Using the notation a_λ one may write rules holding true for all $a,b,c\in\mathcal{Q},\alpha\in\mathbb{R},\lambda,\mu,\nu\in\Lambda$, such as: $(a+b)_\lambda=a_\lambda+b_\lambda$; $(a_\mu+b_\nu)_\lambda=a_{\lambda\mu}+b_{\lambda\nu}$; $(\alpha*c_\mu)_\nu=\alpha*c_{\mu\nu}$, e. g., $(\alpha*c_\mu)_\mu=\alpha*c$. The possibility to perform such symbolic transformations justifies the use of the notation a_- for conjugate instead of the traditional notation $\overline{a}$.

The quasidistributive law. The condition $\alpha\beta \geq 0$ in (40) makes the impression that there may be some freedom in the form of the distributivity relation for $\alpha\beta < 0$. The following result shows that this is not the case: it turns out that (40) determines a specific relation for all $\alpha, \beta \in \mathbb{R}$.

Theorem 1. *(Quasidistributive law) Let $(\mathcal{Q}, +, \mathbb{R}, *)$ be a quasivector space over $\mathbb{R}$. For $\alpha, \beta \in \mathbb{R}$ and $c \in \mathcal{Q}$ we have:*

$$(\alpha + \beta) * c_{\sigma(\alpha+\beta)} = \alpha * c_{\sigma(\alpha)} + \beta * c_{\sigma(\beta)}. \tag{41}$$

Proof. In the case $\sigma(\alpha) = \sigma(\beta)$ (41) is true by assumption (40). Consider the case $\sigma(\alpha) = -\sigma(\beta)$. Assume that $0 \leq \alpha$, $\beta < 0$ and $0 < -\beta \leq \alpha$. In this subcase we have $0 \leq \alpha+\beta$, so that (41) reads: $(\alpha+\beta)*c = a*c+\beta*c_-$. Using (40), we can write

$$\begin{aligned} \alpha * c + \beta * c_- &= ((\alpha + \beta) - \beta) * c + \beta * c_- \\ &= (\alpha + \beta) * c \neg \beta * c + \beta * c_- \\ &= (\alpha + \beta) * c + \beta * (\neg c + c_-) = (\alpha + \beta) * c, \end{aligned}$$

so that (41) is proved to hold true in this subcase. The remaining subcases are verified similarly. □

For $\alpha = 1, \beta = -1$ (41) yields $0 = c \neg c_-$, which is, of course, true. Relation (41) shows that c can be always factored out in an expression of the form $\alpha * c_{\sigma(\alpha)} + \beta * c_{\sigma(\beta)}$. "Dualizing" by $\sigma(\alpha + \beta)$, relation (41) can be written in the equivalent form

$$(\alpha + \beta) * c = \alpha * c_{\sigma(\alpha)\sigma(\alpha+\beta)} + \beta * c_{\sigma(\beta)\sigma(\alpha+\beta)}. \tag{42}$$

Relations (41) and (42) are convenient for symbolic computations. By contrast, without the use of binary variables and the a_λ-notation, formula (42) obtains the form

$$(\alpha + \beta) * c = \begin{cases} \alpha * c + \beta * c, & \text{if } \alpha\beta \geq 0, \\ \alpha * c + \beta * c_-, & \text{if } \alpha\beta < 0,\ |\alpha| \geq |\beta|, \\ \alpha * c_- + \beta * c, & \text{if } \alpha\beta < 0,\ |\alpha| < |\beta|, \end{cases}$$

which can be hardly used for symbolic manipulations.

5 Decomposition of Quasivector Spaces

Definition 5. *Q is a quasivector space. An element $a \in \mathcal{Q}$ with $a \neg a = 0$ is called* linear. *An element $a \in \mathcal{Q}$ with $\neg a = a$ is called* symmetric *or* centred.

It can be easily checked that in a quasivector space $\mathcal{Q}$ the subsets of linear and symmetric elements $\mathcal{Q}' = \{a \in \mathcal{Q} \mid a \neg a = 0\}$, resp. $\mathcal{Q}'' = \{a \in \mathcal{Q} \mid a = \neg a\}$ form subspaces of $\mathcal{Q}$.

Proposition 3. *Assume that Q is a quasivector space. The subspace $\mathcal{Q}' = \{a \in \mathcal{Q} \mid a \neg a = 0\}$ is a vector space.*

Indeed, we only have to check that relation (40) becomes true for all values of the scalars. However, this is obvious from (41).

Definition 6. *Assume that Q is a quasivector space. The space $\mathcal{Q}' = \{a \in \mathcal{Q} \mid a \neg a = 0\}$ is called the* linear subspace *of $\mathcal{Q}$ and the space $\mathcal{Q}'' = \{a \in \mathcal{Q} \mid a = \neg a\}$ is called the* symmetric (centred) subspace *of $\mathcal{Q}$.*

Below we summarize some of the properties of the linear and symmetric elements:

1. $a \in \mathcal{Q}' \Longleftrightarrow a = a_- \Longleftrightarrow a \neg a = 0 \Longleftrightarrow \neg a = \mathrm{opp}(a)$
$$\Longleftrightarrow \exists c \in \mathcal{Q} : a = c + c_-;$$
2. $b \in \mathcal{Q}'' \Longleftrightarrow b = \neg b \Longleftrightarrow b + b_- = 0 \Longleftrightarrow b_- = \mathrm{opp}(b)$
$$\Longleftrightarrow \exists d \in \mathcal{Q} : b = d \neg d.$$

To prove existence, in case 1 take $c = (1/2) * a + s$, where $s \in \mathcal{Q}''$ is arbitrary, and in case 2 take $d = (1/2) * b + t$, where $t \in \mathcal{Q}'$ is arbitrary.

Representation of a quasivector space as a direct sum of a vector and a symmetric subspace. The next theorem shows that every quasivector space is a direct sum of a vector space and a symmetric quasivector space.

Theorem 2. *(***Decomposition theorem***) For every quasivector space $\mathcal{Q}$ we have $\mathcal{Q} = \mathcal{Q}' \oplus \mathcal{Q}''$. More specifically, for every $x \in \mathcal{Q}$ we have $x = x' + x''$ with unique $x' = (1/2) * (x + x_-) \in \mathcal{Q}'$, and $x'' = (1/2) * (x \neg x) \in \mathcal{Q}''$.*

Proof. Assume $x \in \mathcal{Q}$. Using that $x_- \neg x = 0$ we have

$$\begin{aligned} x' + x'' &= (1/2) * (x + x_-) + (1/2) * (x \neg x) \\ &= (1/2) * (x + x + x_- \neg x) = x. \end{aligned}$$

On the other side we have $x' = (1/2) * (x + x_-) \in \mathcal{Q}'$ and $x'' = (1/2) * (x \neg x) \in \mathcal{Q}''$. Hence, $\mathcal{Q} = \mathcal{Q}' + \mathcal{Q}''$. Furthermore, $\mathcal{Q}' \cap \mathcal{Q}'' = 0$. Indeed, assume $x \in \mathcal{Q}'$ and $x \in \mathcal{Q}''$. Then we have simultaneously $x \neg x = 0$ and $x = \neg x$, implying $x = 0$. Hence $\mathcal{Q} = \mathcal{Q}' \oplus \mathcal{Q}''$. □

Theorem 2 states that every element $x \in \mathcal{Q}$ can be decomposed in a unique way as $x' + x''$, where x' is an element of a vector space and x'' belongs to a symmetric quasivector space. We call x' the *linear part (coordinate)* of x, and x'' — the *symmetric (centred) part (coordinate)* of x, and write $x = (x'; x'')$.

In a quasivector space $\mathcal{Q}$ the property $a = \mathrm{opp}(a)$ is satisfied only by the null element 0 of $\mathcal{Q}$. Indeed, $a = \neg a_-$ is equivalent to $a + a = 0$, or $2 * a = 0$, resp. $a = 0$.

Since the distributivity relation is of different form in a vector, resp. symmetric quasivector space, one may wonder how this fact agrees with Theorem 2. Indeed, assume $c \in \mathcal{Q}$ with $c = c' + c''$, $c' \in \mathcal{Q}'$, $c'' \in \mathcal{Q}''$, equivalently: $c = (c'; c'')$. Relation (41) decomposes into

$$\begin{aligned} (\alpha + \beta) * c'_{\sigma(\alpha+\beta)} &= \alpha * c'_{\sigma(\alpha)} + \beta * c'_{\sigma(\beta)}, \\ (\alpha + \beta) * c''_{\sigma(\alpha+\beta)} &= \alpha * c''_{\sigma(\alpha)} + \beta * c''_{\sigma(\beta)}. \end{aligned}$$

Using that c' is linear, we have $c' = c'_-$, so that the first relation is equivalent to the familiar second distributive law: $(\alpha + \beta) * c' = \alpha * c' + \beta * c'$. The second relation retains the form of (41), however, here we should keep in mind that a centred element c'' satisfies $c''_- = \mathrm{opp}(c'')$.

Hints for practical applications. In practice we need to know how to solve problems formulated in quasilinear spaces with monoid structures, like convex bodies and intervals. Assume that $\mathcal{M}$ is a quasilinear space (with monoid structure), cf. Definition 3, and $\mathcal{Q} = \mathcal{D}(\mathcal{M})$ is the induced quasivector space of factorized pairs (A, B), $A, B \in \mathcal{M}$. Let us check how proper elements of $\mathcal{Q}$ of the form $a = (A, 0)$ are decomposed in the form $a = a' + a'' = (a'; a'')$. Are both coordinates a', a'' of a proper element $(a'; a'') \in \mathcal{Q}$ always proper? From Theorem 2 we have $a' = (1/2) * (a +$

$a_-)$, $a'' = (1/2) * (a \neg a)$. Substituting a by $(A, 0)$ and multiplying by 2 we have

$$2 * a' = a + a_- = (A, 0) + (0, \neg A) = (A, \neg A), \quad (43)$$

$$2 * a'' = a \neg a = (A, 0) + (\neg A, 0) = (A \neg A, 0). \quad (44)$$

Relation (44) shows that the symmetric part of a proper element is always proper. However, we see from (43) that the linear part $a' = (A, \neg A)$ of a proper element $(A, 0)$ is not a proper element in general. Next question: when the linear part is a proper element? The element $(A, \neg A)$ is proper, if there exists $X \in \mathcal{M}$, such that $(A, \neg A) = (X, 0)$, that is $A = X \neg A$. For example, in the case of two-dimensional convex bodies, if A is a (proper) triangle, then such X does not exist and consequently the linear part of A is an improper element. In such a situation we need to interpret results which are improper in terms of proper elements.

6 Examples of Quasivector Spaces

Example 1. The system $(\mathcal{K}, +)$ of all convex bodies [25] in a real m-dimensional Euclidean vector space $\mathbb{E}^m$ with set-theoretic (vector, Minkowski) addition: $A + B = \{\alpha + \beta \mid \alpha \in A, \beta \in B\}, A, B \in \mathcal{K}$, is a proper abelian monoid with cancellation law having as a neutral element the origin "0" of $\mathbb{E}^m$. The system $(\mathcal{K}, +, \mathbb{R}, *)$, where "$*$" is the set-theoretic multiplication by real scalars: $\gamma * A = \{\gamma\alpha \mid \alpha \in A\}$, is a quasilinear space (of monoid structure) [19]. To see that the distributive law is violated for convex bodies (and, in particular, for intervals) when $\alpha\beta < 0$, recall that an origin symmetric convex body C satisfies $(-1)*C = C$; hence $1*C + (-1)*C = C + C = 2*C$. On the other side, $(1-1)*C = 0*C = 0$, showing that the second distributive law $(\alpha + \beta) * C = \alpha * C + \beta * C$ is not valid for $\alpha = 1, \beta = -1$ (unless $C = 0$). The monoid $(\mathcal{K}, +)$ induces a group of generalized (extended, directed) convex bodies $(\mathcal{D}(\mathcal{K}), +)$, which has been considered by a number of authors, see e. g. [20]. In [9] we investigate the space $(\mathcal{D}(\mathcal{K}), +, \mathbb{R}, *)$, where "$*$" is defined by (35).

Remark. H. Radström [20] studies the following multiplication by scalars in $(\mathcal{D}(\mathcal{K}), +)$:

$$\gamma \cdot (A, B) = \begin{cases} (\gamma * A, \gamma * B), & \text{if } \gamma \geq 0, \\ (|\gamma| * B, |\gamma| * A), & \text{if } \gamma < 0. \end{cases} \quad (45)$$

If $\gamma < 0$, then $\gamma \cdot (A, 0) = (0, |\gamma| * A)$, which is an improper result. Therefore definition (45) does not provide an extension of multiplication by scalars in

$\mathcal{K}$. Indeed, for $A \in \mathcal{K}$ and $\gamma < 0$, the element $\gamma \cdot A$ should be a proper element, $\gamma \cdot A \in \mathcal{K}$, that is, we should have $\gamma \cdot (A, 0) = \gamma * (A, 0) = (\gamma * A, 0)$, which is not the case. In particular, multiplication by $\gamma = -1$ should produce $(\neg A, 0)$, whereas (45) gives $(0, \neg A)$.

Example 2. A special case of convex bodies are n-dimensional intervals, also called n-dimensional boxes [1], [17], [26], [27]. Given u', $u'' \in \mathbb{R}^n$, $u'' \geq 0$, the set $u = \{\xi \mid u' - u'' \leq \xi \leq u' + u''\}$ is a (compact) n-dimensional *interval* on $\mathbb{R}^n$ with *midpoint (center)* u' and *radius* u'', in *end-point* notation: $u = [u' - u'', u' + u'']$. The *midpoint-radius (center-radius)* presentation an interval $u \in I(\mathbb{R})^n$, briefly called *MR-form*, is $u = (u'; u'')$, where u', $u'' \in \mathbb{R}^n$, $u'' \geq 0$. A common interpretation says that u is an *approximate number (vector)* with a representative value u' and *error bound* u''. The set of all intervals on $\mathbb{R}^n$ is denoted by $I(\mathbb{R})^n$; in particular, for $n = 1$ the set of intervals on $\mathbb{R}$ is denoted by $I(\mathbb{R})$. An n-dimensional interval is an n-tuple of one-dimensional intervals, symbolically $u = (u_1, u_2, ..., u_n)$, $u_i = (u'_i; u''_i) \in I(\mathbb{R}), i = 1, 2, ..., n$. For the midpoint and the radius we have $u' = (u'_1, ..., u'_n),\ u'' = (u''_1, ..., u''_n) \in \mathbb{R}^n$, $u'' \geq 0$, cf. [8], [11], [23], [24].

Consider the interval arithmetic system $(I(\mathbb{R})^n, +, \mathbb{R}, *)$. Addition and multiplication by scalar are defined for $a, b \in I(\mathbb{R})^n$, $\gamma \in \mathbb{R}$, by $a + b = \{\alpha + \beta \mid \alpha \in a, \beta \in b\}$, $\gamma * b = \{\alpha\beta \mid \beta \in b\}$. In MR-form we have:

$$a + b = (a'; a'') + (b'; b'') = (a' + b'; a'' + b''), \tag{46}$$

$$\gamma * b = \gamma * (b'; b'') = (\gamma b'; |\gamma| b''). \tag{47}$$

Multiplication by -1 (negation): $-1 * b = -1 * (b'; b'') = (-b'; b'')$ is briefly denoted $\neg b = -1 * b$. Subtraction

$$a + (\neg b) = a + (-1) * b = (a' - b'; a'' + b'') \tag{48}$$

is briefly denoted as $a \neg b = a + (-1) * b$. We have $a \neg b = \{\alpha - \beta \mid \alpha \in a, \beta \in b\}$.

Remark. The expressions $(a' + b'; a'' + b'')$,- $(\gamma b'; |\gamma| b'')$, $(a' - b'; a'' + b'')$ in the right-hand sides of (46)–(48) are n-dimensional intervals whose midpoints and radii are n-tuples, e. g. $a' + b' = (a'_1 + b'_1, ..., a'_n + b'_n)$, $a'' + b'' = (a''_1 + b''_1, ..., a''_n + b''_n)$, etc. One can also present (46)–(48) first entry-wise in $I(\mathbb{R})^n$, i. e. $a + b = (a_1, a_2, ..., a_n) + (b_1, b_2, ..., b_n) = (a_1 + b_1, a_2 + b_2, ..., a_n + b_n)$, $\alpha * b = \alpha * (b_1, b_2, ..., b_n) = (\alpha * b_1, \alpha * b_2, ..., \alpha * b_n)$, $a \neg b = (a_1 \neg b_1, a_2 \neg b_1, ..., a_n \neg b_n)$, and then using formulae (46)–(48) for the one-dimensional components, namely $a_i + b_i = (a_i' + b_i'; a''_i + b''_i)$, $\alpha * b_i = (\alpha * b_i'; |\alpha| * b''_i)$, $a_i \neg b_i = (a_i' - b_i'; a''_i + b''_i)$, $i = 1, ..., n$.

The presentation $a = (a'; 0) + (0; a'')$ shows that every interval is a sum of a *point interval* of the form $(u; 0)$ and a *centred (origin-symmetric) interval* of the form $(0; v), v \geq 0$.

The interval arithmetic system $(I(\mathbb{R})^n, +, \mathbb{R}, *)$ is a quasilinear space (of monoid structure) [5], [6], [16], [21], [22]. This space induces a quasivector space $(D(I(\mathbb{R})^n), +, \mathbb{R}, *)$ of generalized/directed intervals [4], [10]. We have $D(I(\mathbb{R})^n) = \mathbb{V}^n \oplus \mathbb{S}^n$, where $\mathbb{V}^n = (\mathbb{R}^n, +, \mathbb{R}, \cdot)$, $n \geq 1$ is the familiar linear (vector) space under the operations of addition and multiplication by scalars, and $\mathbb{S}^n$ is the symmetric quasivector space presented in the next Example.

Example 3. For any integer $k \geq 1$ the set $\mathbb{R}^k$ of all k-tuples $(\alpha_1, \alpha_2, ..., \alpha_k)$, where $\alpha_i \in \mathbb{R}$ and $(\alpha_1, \alpha_2, ..., \alpha_k) = (\beta_1, \beta_2, ..., \beta_k)$ if and only if $\alpha_1 = \beta_1, \alpha_2 = \beta_2, ..., \alpha_k = \beta_k$, forms a quasivector space over $\mathbb{R}$ under the following operations

$$\begin{aligned} (\alpha_1, \alpha_2, ..., \alpha_k) + (\beta_1, \beta_2, ..., \beta_k) &= (\alpha_1 + \beta_1, \alpha_2 + \beta_2, ..., \alpha_k + \beta_k), \quad (49) \\ \gamma * (\alpha_1, \alpha_2, ..., \alpha_k) &= (|\gamma|\alpha_1, |\gamma|\alpha_2, ..., |\gamma|\alpha_k), \ \gamma \in \mathbb{R}. \quad (50) \end{aligned}$$

This quasivector space is denoted by $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$ and called *canonical symmetric quasivector space* (of dimension k). Note that multiplication by -1 (negation) in $\mathbb{S}^k$ is same as identity while the opposite operator is same as conjugation:

$$\mathrm{opp}(\alpha_1, \alpha_2, ..., \alpha_k) = (\alpha_1, \alpha_2, ..., \alpha_k)_- = (-\alpha_1, -\alpha_2, ..., -\alpha_k).$$

Denoting $\mathbb{S} = \mathbb{S}^1$, we have $\mathbb{S}^k = \mathbb{S} \oplus \mathbb{S} \oplus \ldots \oplus \mathbb{S}$.

Example 4. Consider the set of infinite sequences $(\alpha_1, \alpha_2, ...)$, $\alpha_i \in \mathbb{R}$, with addition and multiplication by scalars defined as in (49) and (50), i. e.:

$$\begin{aligned} (\alpha_1, \alpha_2, ...) + (\beta_1, \beta_2, ...) &= (\alpha_1 + \beta_1, \alpha_2 + \beta_2, ...), \\ \gamma * (\alpha_1, \alpha_2, ...) &= (|\gamma|\alpha_1, |\gamma|\alpha_2, ...), \ \ \gamma \in \mathbb{R}. \end{aligned}$$

We again obtain a quasivector space.

Example 5. Consider the direct sum $\mathbb{V}^l \oplus \mathbb{S}^k$ of the l-dimensional vector space $\mathbb{V}^l = (\mathbb{R}^l, +, \mathbb{R}, \cdot)$ and the quasivector space $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$ from Example 3. The elements of $\mathbb{V}^l \oplus \mathbb{S}^k$ are n-tuples, $n = l + k$, of the form $(\lambda_1, ..., \lambda_l; \lambda_{l+1}, ..., \lambda_{l+k})$. Addition and multiplication by scalars ($\gamma \in \mathbb{R}$) are:

$$\begin{aligned} &(\lambda_1, ..., \lambda_l; \lambda_{l+1}, ..., \lambda_{l+k}) + (\mu_1, ..., \mu_l; \mu_{l+1}, ..., \mu_{l+k}) \\ &\qquad = (\lambda_1 + \mu_1, ..., \lambda_l + \mu_l; \lambda_{l+1} + \mu_{l+1}, ..., \lambda_{l+k} + \mu_{l+k}), \\ &\gamma * (\lambda_1, ..., \lambda_l; \lambda_{l+1}, ..., \lambda_{l+k}) = (\gamma\lambda_1, ..., \gamma\lambda_l; |\gamma|\lambda_{l+1}, ..., |\gamma|\lambda_{l+k}). \end{aligned}$$

As direct sum of two quasivector spaces, $\mathbb{V}^l \oplus \mathbb{S}^k$ is a quasivector space. Negation

$$(-1) * (\lambda_1, ..., \lambda_l; \lambda_{l+1}, ..., \lambda_{l+k}) = (-\lambda_1, ..., -\lambda_l; \lambda_{l+1}, ..., \lambda_{l+k})$$

is distinct from opposite: $\mathrm{opp}(\lambda_1, ..., \lambda_{l+k}) = (-\lambda_1, ..., -\lambda_l; -\lambda_{l+1}, ..., -\lambda_{l+k})$. The composition of the opposite and negation operators yields:

$$\mathrm{opp}(\neg(\lambda_1, ..., \lambda_l;\ \lambda_{l+1}, ..., \lambda_{l+k})) = (\lambda_1, ..., \lambda_l; -\lambda_{l+1}, ..., -\lambda_{l+k}).$$

The space $\mathbb{V}^l \oplus \mathbb{S}^k$ can be used for an explicite presentation and computation with zonotopes, see e. g. [14], [15]. For $l = k$ we have the space of boxes considered in Example 2.

Example 6. The set of all real functions is a quasivector space if we define $f + g$ as the function whose value at x is $f(x) + g(x)$, and $\gamma * f$ as a function whose value at x is

$$\gamma * f(x) = \begin{cases} \gamma \cdot f(x), & \text{if } \gamma \geq 0, \\ |\gamma| \cdot f(-x), & \text{if } \gamma < 0. \end{cases} \tag{51}$$

In particular, negation is: $-1 * f(x) = f(-x)$. Note that in this quasivector space negation is distinct from opposite $\mathrm{opp}(f) = -f$. Note that the composition of opposite and negation $-f(-x)$ is a new operator. The operation (51) appears in the theory of (differences of) support functions, cf. [9]. We note that, if f is the support function of $A \in \mathcal{K}$, then (51) is the support function of the convex body $\gamma * A$; in particular, $-1 * f(x) = f(-x)$ is the support function of $\neg A$. The symmetric (centred) functions are those satisfying $f(x) = f(-x)$ and the linear ones are those with the property $g(x) + g(-x) = 0$, that is $-g(x) = g(-x)$.

Example 7. Let $\mathbb{C} = (\mathbb{C}, +, \mathbb{R}, \cdot)$ be the vector space of all complex numbers $c = c_1 + ic_2$ with addition: $(c_1+ic_2)+(d_1+id_2) = (c_1+d_1)+i(c_2+d_2)$ and multiplication by real scalars: $\gamma \cdot (c_1 + ic_2) = \gamma c_1 + i\gamma c_2$. Opposite is $-c = -c_1 - ic_2$. Another important involution in $\mathbb{C}$ is conjugation. We shall show that, when equipped with conjugation, $\mathbb{C}$ is a quasivector space. Conjugate elements are introduced in $\mathbb{C}$ by means of: $c_- = \overline{c} = c_1 - ic_2$; in particular $\overline{i} = -i$. Define a new "quasivector" multiplication by scalars in $\mathbb{C}$ by:

$$\gamma * c = \begin{cases} \gamma \cdot c, & \text{if } \gamma \geq 0, \\ \gamma \cdot \overline{c}, & \text{if } \gamma < 0. \end{cases}$$

Negation in $\mathbb{C}^*$ is $\neg c = (-1) * c = -\overline{c} = -(c_1 - ic_2) = -c_1 + ic_2$. The system $\mathbb{C}^* = (\mathbb{C}, +, \mathbb{R}, *)$ is a quasivector space. Indeed, using the notation $c_- = \overline{c}$, $c_+ = c$, it is easy to show that relation (41) holds. We have

$\mathbb{C}^* = \mathbb{V}^1 \oplus \mathbf{Im}$, where $\mathbf{Im} = (\mathbf{Im}, +, \mathbb{R}, *)$ is the symmetric quasivector space of purely imaginary numbers. Note that in **Im** negation is same as identity, whereas conjugation is same as opposite. Note that the "quasivector" multiplication "$*$" of a complex number c by scalars does not change the sign of the imaginary part c_2, whereas the "linear" multiplication "$\cdot$" may change it (whenever the scalar is negative).

7 Linear and quasivector operations

Let $(\mathcal{Q}, +, \mathbb{R}, *)$ be a quasivector space over the l. o. field $\mathbb{R}$. Certain operations in $\mathcal{Q}$ are specific for linear spaces and can be characterized as linear, whereas other operations are specific only for quasivector spaces and can be characterized as quasivector ones.

Consider the operation "$\cdot$": $\mathbb{R} \times \mathcal{Q} \longrightarrow \mathcal{Q}$ defined for $\alpha \in \mathbb{R},\ c \in \mathcal{Q}$ by

$$\alpha \cdot c = \alpha * c_{\sigma(\alpha)} = \begin{cases} \alpha * c, & \text{if } \alpha \geq 0, \\ \alpha * c_-, & \text{if } \alpha < 0. \end{cases} \tag{52}$$

Theorem 3. *Let $(\mathcal{Q}, +, \mathbb{R}, *)$ be a quasivector space over $\mathbb{R}$. Then $(\mathcal{Q}, +, \mathbb{R}, \cdot)$, with "$\cdot$" defined by (52), is a vector space over $\mathbb{R}$.*

Proof. Indeed, we need to check that the distributive law holds true. This is obvious if we substitute all terms in the quasidistributive law (41): $(\alpha + \beta) * c_{\sigma(\alpha+\beta)} = \alpha * c_{\sigma(\alpha)} + \beta * c_{\sigma(\beta)}$ by the corresponding linear terms using relation (52), that is $\alpha * c_{\sigma(\alpha)} = \alpha \cdot c$, $\beta * c_{\sigma(\beta)} = \beta \cdot c$, $(\alpha+\beta) * c_{\sigma(\alpha+\beta)} = (\alpha + \beta) \cdot c$, obtaining thus the distributive law $(\alpha + \beta) \cdot c = \alpha \cdot c + \beta \cdot c$. Checking the rest of the relations of a linear space is trivial. □

Operation (52) is well defined on $\mathbb{R} \times \mathcal{Q}$ for any quasivector space $\mathcal{Q}$ over $\mathbb{R}$; it is called *linear multiplication* in $\mathcal{Q}$. By contrast, the original multiplication "$*$" in $\mathbb{R} \times \mathcal{Q}$ is further called *quasivector multiplication.* Theorem 3 implies that every quasivector space $(\mathcal{Q}, +, \mathbb{R}, *)$ involves a linear multiplication and hence an *associated vector space* $(\mathcal{Q}, +, \mathbb{R}, \cdot)$.

Remark. Note that the associated vector space $(\mathcal{Q}, +, \mathbb{R}, \cdot)$ is not the linear subspace $\mathcal{Q}'$ from the decomposition theorem; there $\mathcal{Q}'$ consists just of the linear elements of $\mathcal{Q}$.

Relation (52) shows that the linear multiplication is an inherent operation for every quasilinear space; although it is not explicitly included in the notation $(\mathcal{Q}, +, \mathbb{R}, *)$, it is present there in the same way as the operators opposite, negation and conjugation are present. For the sake of easy reference in the sequel we shall denote the three operations: addition "+", opposite "opp" and linear multiplication by scalar "·" as *linear operations* whereas the operation quasivector multiplication by scalars "$*$" will be denoted as quasivector (or quasilinear) operation. To this operation we shall add the operator negation "$\neg$" as special case of the quasivector multiplication by scalars "$*$". Also we should add the operation conjugation "c_-" as composition of opposite and negation. To summarize, "$*$", "$\neg$" and "c_-" are quasilinear operations. Any composition or expression involving them will also be denoted as quasilinear.

Remark. Special cases of linear multiplication have been considered by Radstroem [20] and Gardenes [2]–[3]. It is easy to see that Radstroem formula (45) is a consequence of relation (52).

The question arises: is it possible to represent all quasilinear operations and expressions in a quasivector space by means of linear operations?

The answer to the above question is generally "no". For we cannot generally represent in linear terms the quasivector multiplication by scalars. Indeed, from (52) we have

$$\alpha * c = \alpha \cdot c_{\sigma(\alpha)} = \begin{cases} \alpha \cdot c, & \text{if } \alpha \geq 0, \\ \alpha \cdot c_-, & \text{if } \alpha < 0. \end{cases} \tag{53}$$

Relation (53) shows that the quasivector expression $\alpha * c$ is presented as $\alpha \cdot c_{\sigma(\alpha)}$ using the linear multiplication **and** the **quasivector** operator conjugation. Clearly, the quasivector operations "$*$", "$\neg$" and "c_-" cannot be generally represented by means of linear operations. However, in the sequel we are going to show that such a representation does exist in the special case of symmetric quasivector spaces.

Up to now when defining a quasivector space we first introduced a quasilinear multiplication by scalars, then the operator conjugation is defined as composition of opposite and negation (which is multiplication by -1). Next we show that it is possible to proceed in inverse order: to introduce first an operator with the properties of conjugation and then to define multiplication by scalars using this operator.

Vector spaces with involution involve quasivector spaces. A linear transformation $i : \mathcal{G} \longrightarrow \mathcal{G}$, such that for $a, b \in \mathcal{G}$, $\alpha \in \mathbb{R}$: 1) $i(a+b) = i(a) + i(b)$, 2) $i(\alpha \cdot c) = \alpha \cdot i(c)$, satisfying the additional assumption: 3) $i^2(a) = a$, is called an *involution (dual automorphism, symmetry)* in $\mathcal{G}$. Note that an involution also satisfies: $i(a) = 0$ iff $a = 0$. In every vector space $(\mathcal{G}, +, \mathbb{R}, \cdot)$, there are (at least) two involutions: identity and opposite.

Assume that $\mathcal{G} = (\mathcal{G}, +, \mathbb{R}, \cdot)$ is a vector space over the l. o. field $\mathbb{R}$ and let i be an involution in $\mathcal{G}$. Define $* : \mathbb{R} \times \mathcal{G} \longrightarrow \mathcal{G}$ by

$$\alpha * c = \begin{cases} \alpha \cdot c, & \text{if } \alpha \geq 0, \\ \alpha \cdot i(c), & \text{if } \alpha < 0. \end{cases} \tag{54}$$

Theorem 4. *Let $\mathcal{G} = (\mathcal{G}, +, \mathbb{R}, \cdot, i)$ be a vector space (over the l. o. field $\mathbb{R}$) equipped with an involution i. Let $* : \mathbb{R} \times \mathcal{G} \longrightarrow \mathcal{G}$ be defined by (54). Then the system $(\mathcal{G}, +, \mathbb{R}, *)$ induced by $(\mathcal{G}, +, \mathbb{R}, \cdot, i)$ is a quasivector space.*

If we substitute the involution i in (54) by identity $i(c) = c$, we obtain $\alpha * c = \alpha \cdot c$ for all $\alpha \in \mathbb{R}$, $c \in \mathcal{G}$, that is both multiplications "$\cdot$" and "$*$" coincide; this is a trivial case, as the induced quasivector space coincides with the original vector space. If we choose the involution in the vector space to be opposite, $i(c) = \text{opp}(c)$, then (54) obtains the form $\alpha * c = |\alpha| \cdot c$. Setting $\alpha = -1$ yields $\neg c = c$ meaning that all elements are centred. This case is discussed in the next section.

Setting $\alpha = -1$ in (54) we obtain $(-1) * c = (-1) \cdot i(c) = \text{opp}(i(c))$. Thus i coincides with conjugation: $i(c) = \text{opp}(\neg c) = c_-$, showing that formula (53) is same as formula (54).

According to Theorem 4 every vector space equipped with involution i (called conjugation) generates a quasivector space. In the special case, when the involution coincides with some of the (two) involutions in the vector space (identity or opposite), then the quasivector space is either linear or symmetric. If i is neither identity nor opposite, then the quasivector space has (at least) four involutions: identity, opposite, conjugation (i) and negation $\text{opp}(i)$, which is a composition of opposite and conjugation. In this latter case the quasivector space can be decomposed as a direct sum of a linear and a symmetric quasivector space using the decomposition theorem.

8 Symmetric Quasivector Spaces

The decomposition Theorem 2 states that every quasivector space is a direct sum of a vector space and a symmetric quasivector space. As vector spaces are well-known, we need to study symmetric quasivector spaces.

A symmetric quasivector space $\mathcal{S}$ can be defined axiomatically as an abelian group with multiplication by scalars (from a l. o. field) satisfying (37)–(40) together with the additional assumption: $\neg a = a$ for all $a \in \mathcal{S}$. Starting from this axiomatic definition one can develop the theory of symmetric quasivector spaces following step by step the development of the theory of vector spaces. This approach has been exploited in [9] and has been sketched in the Appendix. In this section we concentrate on the relations between quasivector and linear multiplication by scalars; then the properties of symmetric quasivector spaces become transparent.

The symmetric case. In a symmetric quasivector space, due to $(-1) * c = c$, we have: $\alpha * c = (-\alpha) * c = |\alpha| * c$. Hence formula (52) for the induced linear multiplication in a symmetric quasivector space can be written as

$$\alpha \cdot c = |\alpha| * c_{\sigma(\alpha)}. \tag{55}$$

Recall the relation opp $(c) = c_-$, which is true for symmetric elements and not true in a general quasivector space. This relation shows that, in the case of symmetric quasivector space, the operation conjugation can be presented by the linear operation "opposite". Substituting $c_- = \text{opp}(c)$ in (53) gives

$$\alpha * c = |\alpha| \cdot c. \tag{56}$$

Relation (56) shows that in a symmetric quasivector space the quasivector multiplication "$*$" is representable by linear operations. In particular, negation is representable by linear operations, due to $\neg c = c$. (Note again that the above is not true in a general quasivector space, but is true in a symmetric quasivector space.) We thus obtain the following special case of Theorem 3:

Corollary 1. Let $(\mathcal{S}, +, \mathbb{R}, *)$ be a symmetric quasivector space over $\mathbb{R}$. The induced vector space $(\mathcal{S}, +, \mathbb{R}, \cdot)$, with "$\cdot$" defined by (55), due to (56), involves implicitly quasilinear multiplication by scalars, and hence all quasilinear operations/expressions can be presented by linear ones.

We now formulate the special case of Theorem 4 concerning centred elements.

Corollary 2. Every vector space over a l. o. field $\mathbb{R}$ induces via (56) a symmetric quasivector space.

Thus to every vector space over a l. o. field $(\mathcal{G}, +, \mathbb{R}, \cdot)$, we associate the symmetric quasivector space $(\mathcal{G}, +, \mathbb{R}, *)$ with "$*$" defined by (56). The two spaces — $(\mathcal{Q}, +, \mathbb{R}, *)$ and $(\mathcal{Q}, +, \mathbb{R}, \cdot)$ — differ from each other by having different operations for multiplication by scalars.

The "symmetric" case can be summarized as follows: *Every symmetric quasivector space over* $\mathbb{R}$ *generates via (55) a lienear multiplication by scalars* *and hence a vector space. Vice versa, every vector space over $\mathbb{R}$ induces via (56) a quasilinear multiplication by scalars and thus a symmetric quasivector space.*

The above allows us to look at the spaces $(\mathcal{S}, +, \mathbb{R}, *)$ and $(\mathcal{S}, +, \mathbb{R}, \cdot)$ as at one and the same space with two different operations for multiplication by scalars: $(\mathcal{S}, +, \mathbb{R}, *, \cdot)$. Let $(\mathcal{S}, +, \mathbb{R}, *)$ be a *symmetric* quasivector space and $(\mathcal{S}, +, \mathbb{R}, \cdot)$ be the associated vector space. It follows from the above, that all concepts characteristic for the vector space $(\mathcal{S}, +, \mathbb{R}, \cdot)$, such as linear combination, linear dependence, basis etc., can be represented in terms of the original symmetric quasivector space $(\mathcal{S}, +, \mathbb{R}, *)$. We thus feel free to use any vector space concepts in the symmetric quasivector space.

For example, using (52) we can reformulate the familiar linear combination $f = \sum_{i=1}^{k} \alpha_i \cdot c^{(i)} = \alpha_1 \cdot c^{(1)} + \alpha_2 \cdot c^{(2)} + ... + \alpha_k \cdot c^{(k)}$ in quasivector terms to obtain:

$$f = \alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(2)}_{\sigma(\alpha_2)} + ... + \alpha_k * c^{(k)}_{\sigma(\alpha_k)}.$$

For more detail and a self-contained exposition of the theory of symmetric quasivector spaces, see Appendix.

Exercise. Use (56) to define the symmetric quasilinear space corresponding to the vector space $\mathbb{R}^n$ and show that this is the space $\mathbb{S}^n$ defined in Example 3. Use relation (55) to define the vector space corresponding to the symmetric quasivector space $\mathbb{S}^n$ and show that this is the space $\mathbb{R}^n$.

As every quasivector space $\mathcal{Q}$ is a direct sum $\mathcal{Q} = \mathcal{V} \oplus \mathcal{S}$ of a vector space $\mathcal{V}$ and a symmetric quasivector space $\mathcal{S}$, we can speak of basis and dimension of $\mathcal{Q}$, whenever $\mathcal{V}$ and $\mathcal{S}$ have finite bases. Namely, let $\mathcal{V} = \mathcal{V}^l$ be a l-dimensional vector space with a basis $(v^{(1)}, ..., v^{(l)})$ and let $\mathcal{S} = \mathbb{S}^k$ be a k-dimensional symmetric quasivector space having a basis $(s^{(1)}, ..., s^{(k)})$. Then we say that $(v^{(1)}, ..., v^{(l)}; s^{(1)}, ..., s^{(k)})$ is a basis of the (l, k)-*dimensional* quasivector space $\mathcal{Q} = \mathbb{V}^l \oplus \mathbb{S}^k$.

9 Appendix. Symmetric quasilinear spaces: a self-contained approach

Linear combinations in symmetric quasivector spaces. Assume that $(\mathcal{S}, +, \mathbb{R}, *)$ is a *symmetric* quasivector space. Now all vector space concepts that are characteristic in the related vector space, such as linear combination, linear dependence, basis etc., can be represented in terms of the original symmetric quasivector space $(\mathcal{S}, +, \mathbb{R}, *)$. For example, the concept of linear combination obtains the following form.

Let $c^{(1)}, c^{(2)}, ..., c^{(k)}$ be finitely many (not necessarily distinct) elements of $\mathcal{S}$. An element $f \in \mathcal{S}$ of the form

$$f = \alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(2)}_{\sigma(\alpha_2)} + ... + \alpha_k * c^{(k)}_{\sigma(\alpha_k)}, \tag{57}$$

where $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{R}$, is called a *linear combination* of $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{S}$.

Proposition 4. *Let $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{S}$, $k \geq 1$. Then the set*

$$\mathcal{H} = \{\sum_{i=1}^{k} \alpha_i * c^{(i)}_{\sigma(\alpha_i)} \mid \alpha_i \in \mathbb{R},\ i = 1, ..., k\}$$

of all linear combinations of $c^{(1)}, c^{(2)}, ..., c^{(k)}$ is a subspace of $\mathcal{S}$.

The proof is elementary — it can be done by translating the corresponding theorem from the theory of vector spaces in quasivector terminology.

In order to demonstrate the selfsufficiency of the theory of quasivector spaces, in the remaining part of this Appendix we present some basic notions of quasivector spaces.

The elements $c^{(1)}, c^{(2)}, ..., c^{(k)}$ form a *generating set* for $\mathcal{H}$. We also say that the subspace $\mathcal{H}$ defined in Proposition 4 is *spanned* by $c^{(1)}, c^{(2)}, ..., c^{(k)}$ and write $\mathcal{H} = \text{span}\{c^{(1)}, c^{(2)}, ..., c^{(k)}\}$.

Let $\mathcal{S}$ be a symmetric quasivector space over $\mathbb{R}$. The elements $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{S}$, $k \geq 1$, are *linearly dependent (over $\mathbb{R}$)*, if there exists a nontrivial linear combination of $\{c^{(i)}\}$, which is equal to 0, i. e. if there exists a system $\{\alpha_i\}_{i=1}^{k}$ with not all α_i equal to zero, such that

$$\alpha_1 * c^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * c^{(1)}_{\sigma(\alpha_2)} + ... + \alpha_k * c^{(k)}_{\sigma(\alpha_k)} = 0. \tag{58}$$

Elements of $\mathcal{S}$, which are not linearly dependent, are *linearly independent*. That is, the elements $c^{(1)}, c^{(2)}, ..., c^{(k)} \in \mathcal{S}$ are *linearly independent*, if (58) is possible only for the trivial linear combination, such that $\alpha_i = 0$ for all $i = 1, ..., k$.

Linear mappings in quasivector spaces. Let $\mathcal{Q}_1 = (\mathcal{Q}_1, +, \mathbb{R}, *)$, $\mathcal{Q}_2 = (\mathcal{Q}_2, +, \mathbb{R}, *)$ be two quasivector spaces over $\mathbb{R}$ and let $\varphi : \mathcal{Q}_1 \longrightarrow \mathcal{Q}_2$ be a homomorphic (linear) mapping, that is:

$$\varphi(x+y) = \varphi(x) + \varphi(y), \tag{59}$$

$$\varphi(\lambda * x) = \lambda * \varphi(x), \quad x, y \in \mathcal{Q}_1, \ \lambda \in \mathbb{R}. \tag{60}$$

It is easy to check that $\varphi(x_-) = (\varphi(x))_-$; more generally any linear mapping satisfies:

$$\varphi(\alpha_1 * x^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * x^{(2)}_{\sigma(\alpha_2)} + ... + \alpha_k * x^{(k)}_{\sigma(\alpha_k)}) = \tag{61}$$
$$\alpha_1 * \varphi(x^{(1)})_{\sigma(\alpha_1)} + \alpha_2 * \varphi(x^{(2)})_{\sigma(\alpha_2)} + ... + \alpha_k * \varphi(x^{(k)})_{\sigma(\alpha_k)},$$

where $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{R}$, $x^{(1)}, x^{(2)}, ..., x^{(k)} \in \mathcal{Q}_1$. In particular:

$$\varphi(\alpha * x_\lambda + \beta * y_\mu) = \alpha * \varphi(x)_\lambda + \beta * \varphi(y)_\mu, \ x, y \in \mathcal{Q}_1, \ \lambda, \mu \in \mathbb{R}. \tag{62}$$

Condition (62) completely characterizes a linear mapping and can substitute conditions (59) and (60).

Let $\mathcal{S}$ be a symmetric quasivector space and $x^{(1)}, x^{(2)}, ..., x^{(n)} \in \mathcal{S}$ and let $\mathbb{S}^n = (\mathbb{R}^n, +, \mathbb{R}, *)$ be the canonic symmetric quasivector space defined in Example 3. It can be easily checked that the mapping $\varphi : \mathbb{S}^n \longrightarrow \mathcal{S}$ such that

$$\varphi(\alpha_1, \alpha_2, ..., \alpha_n) = \alpha_1 * x^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * x^{(2)}_{\sigma(\alpha_2)} + ... + \alpha_n * x^{(n)}_{\sigma(\alpha_n)} \tag{63}$$

is linear.

Denote $e^{(i)} = (0, 0, ..., 0, 1, 0, ..., 0)$, where the component 1 is on the i-th place. We consider $e^{(i)}$ as elements of $\mathbb{S}^n$, where $\mathrm{opp}(e^{(i)}) = e^{(i)}_-$ and $\neg e^{(i)} = e^{(i)}$. Relation (63) implies

$$\varphi(e^{(i)}) = \alpha_i * x^{(i)}_{\sigma(\alpha_i)}|_{\alpha_i = 1} = x^{(i)}, \ i = 1, ..., n. \tag{64}$$

The mapping φ is the only linear mapping from $\mathbb{S}^n$ to $\mathcal{S}$ with the property (64). Indeed, if (64) holds, then by (61),

$$\begin{aligned}\varphi(\alpha_1, \alpha_2, ..., \alpha_n) &= \varphi(\sum \alpha_i * e^{(i)}_{\sigma(\alpha_i)}) \\ &= \sum \alpha_i * \varphi(e^{(i)})_{\sigma(\alpha_i)} = \sum \alpha_i * x^{(i)}_{\sigma(\alpha_i)}.\end{aligned}$$

We thus obtain that relation (64): $\varphi(e^{(i)}) = x^{(i)}, \ i = 1, ..., n$, is sufficient to determine the mapping (63). As in the case of vector spaces, every mapping of the set $(e^{(1)}, ..., e^{(n)})$ into $\mathcal{S}$ of the form $\varphi(e^{(i)}) = x^{(i)},\ i = 1, ..., n$, can be extended to a unique linear mapping of $\mathbb{S}^n$ into $\mathcal{S}$.

Basis in a symmetric quasivector space. Let $\mathcal{S}$ be a symmetric quasivector space over $\mathbb{R}$. The set $\{c^{(i)}\}_{i=1}^k$, $c^{(i)} \in \mathcal{S}$, $k \geq 1$, is a *basis* of $\mathcal{S}$, if $c^{(i)}$ are linearly independent and $\mathcal{S} = \text{span}\{c^{(i)}\}_{i=1}^k$.

Let $\mathcal{S}$ be a symmetric quasivector space over $\mathbb{R}$ and $\{c^{(i)}\}_{i=1}^k$ be a basis of $\mathcal{S}$. Assume that $a = \sum_{i=1}^k \alpha_i * c^{(i)}_{\sigma(\alpha_i)},\ \ b = \sum_{i=1}^k \beta_i * c^{(i)}_{\sigma(\beta_i)}$ are two elements of $\mathcal{S}$. Their sum is

$$a + b = \sum_{i=1}^{k} \alpha_i * c^{(i)}_{\sigma(\alpha_i)} + \sum_{i=1}^{k} \beta_i * c^{(i)}_{\sigma(\beta_i)} = \sum_{i=1}^{k} (\alpha_i + \beta_i) * c^{(i)}_{\sigma(\alpha_i+\beta_i)}. \tag{65}$$

Multiplication by scalars is given by

$$\gamma * a = \sum_{i=1}^{k} |\gamma|\alpha_i * c^{(i)}_{\sigma(\alpha_i)} = \sum_{i=1}^{k} |\gamma|\alpha_i * c^{(i)}_{\sigma(|\gamma|\alpha_i)}. \tag{66}$$

To every $a = \sum_{i=1}^k \alpha_i * c^{(i)}_{\sigma(\alpha_i)} \in \mathcal{S}$ we associate the vector $(\alpha_1, \alpha_2, ..., \alpha_k)$. Then, minding formulae (65), (66), we define addition and multiplication by scalars by means of (49), (50), arriving thus to the canonic symmetric quasivector space $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$ considered in Example 2.

Theorem 5. *Any symmetric quasivector space over the l. o. field of reals $\mathbb{R}$, with a basis of k elements, is isomorphic to $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$.*

Proof. Let $\mathcal{S}$ be a symmetric quasivector space spanned over a finite basis $s^{(1)}, s^{(2)}, ..., s^{(k)}$. The linear mapping $\varphi : \mathbb{S}^k \longrightarrow \mathcal{S}$, $\mathbb{S}^k = (\mathbb{R}^k, +, \mathbb{R}, *)$, defined by

$$\varphi(\alpha_1, \alpha_2, ..., \alpha_k) = \alpha_1 * s^{(1)}_{\sigma(\alpha_1)} + \alpha_2 * s^{(2)}_{\sigma(\alpha_2)} + ... + \alpha_k * s^{(k)}_{\sigma(\alpha_k)},$$

is a bijection. Hence φ is an isomorphism. $\square$

Let $\mathcal{S}$ be a symmetric quasivector space spanned over a finite basis $s^{(1)}, s^{(2)}, ..., s^{(k)}$. Clearly, as in the linear case, the number k of terms in the expression for the span does not change with the particular basis, hence will be called *dimension* of $\mathcal{S}$.

10 Concluding Remarks

In this paper we present a novel approach in the exposition of the theory of quasivector spaces (with group structure), stressing on the relation between quasivector and vector spaces. We show that every quasivector space is decomposed into a vector space and a symmetric quasivector space. We demonstrate that the latter space can be turned into a vector space by a redefinition of the operation multiplication by scalars. Our investigations show the theoretical and practical importance of the midpoint-radius representation of intervals, already noticed in the pioneering papers [26], [27]; similar representation is used in e. g. [23], [24].

Acknowledgements. This work has been accomplished within the project “Fundamentals of Computational Mathematics” supported by the Brazilian Science Agencies CTINFO/CNPq and FAPERGS and has been partially supported by the Bulgarian National Science Fund under grant No. IS-03/2003.

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