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Extended Interval Arithmetic and some Applications

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1 Introduction

There are four familiar (standard) arithmetic operations between compact intervals used in interval arithmetic [10, 17]. In addition to them we shall use four other arithmetic operations, which we shall call *non-standard*. Thus we shall use eight arithmetic operations alltogether. However, the classification of the eight arithmetic operations into standard and non-standard is not very convenient. We shall classify the whole set of operations into two groups — four *basic* arithmetic operations and four *auxiliary* arithmetic operations, which are compositions of the basic ones. The set of basic operations comprises two standard operations (the familiar addition and multiplication) and two non-standard operations (for subtraction and division). In this paper we consider the algebraic structure of the space of compact intervals together with the basic arithmetic operations and give some possible applications of this structure.

We shall give first a general idea of the basic and auxiliary operations and some motivation for their use. More motivations will be given in Sections 5 and 8.

The standard arithmetic addition and multiplication of intervals $a = [a_1, a_2]$, $b = [b_1, b_2]$ are defined by $a * b = \{\alpha * \beta \mid \alpha \in a,\ \beta \in b\}$, where $* \in \{+, \times\}$. We may note that the interval $a * b$ is the largest interval determined by the points $a_i * b_i$, $i, j = 1, 2$, that is

$$a * b = [\min\{a_1 * b_1, a_1 * b_2, a_2 * b_1, a_2 * b_2\}, \max\{a_1 * b_1, a_1 * b_2, a_2 * b_1, a_2 * b_2\}].$$

Denote the (actual) end-points of this interval by α and β, that is $a * b = [\alpha, \beta]$, where $\alpha, \beta \in \{a_i * b_i\}_{i,j=1,2}$. Assume that a and b are not degenerate, that is $a_1 \neq a_2$, $b_1 \neq b_2$, so that $a_i * b_i$, $i, j = 1, 2$, are four different points and exclude the points α, β from the set $\{a_i * b_i\}$. The remaining two points then, say γ, δ, $\gamma < \delta$, determine a shorter interval $[\gamma, \delta]$, such that $[\gamma, \delta] \in [\alpha, \beta]$. We shall denote this interval by $a \oplus b$ or $a \otimes b$ depending on whether $* = +$ or $* = \times$. In interval computations we shall sometimes make use of this interval(s).

To illustrate this let us express the interval $c = \{\alpha + \alpha^2 \mid \alpha \in a\}$ by means of the interval a, where a does not intersect $(-1/2, 0)$. If $a \geq 0$, we can write $c = a + a^2$, where $a^2 = a \times a$. If $a \leq -1/2$, then c cannot be expressed by a using standard interval arithmetic. However, we can write $c = a \oplus a^2$ in this case, using auxiliary addition. The operations $a \oplus b$ and $a \otimes b$ are defined for arbitrary intervals a, b in sections 1 and 6 respectively.

The standard arithmetic subtraction and division are usually denoted by "$-$" and "$/$" resp. However, it will be more convenient for us to denote these operations by $\ominus$ and $\oslash$ resp., that is $a \ominus b = \{\alpha - \beta \mid \alpha \in a,\ \beta \in b\}$, $a \oslash b = \{\alpha/\beta \mid \alpha \in a,\ \beta \in b\}$. Thereby we preserve the notations $a - b$ and a/b for non-standard operations.

Note that the interval $a \ominus b$ is the largest interval determined by the points $a_i - b_j$, $i, j = 1, 2$. Suppose again for simplicity that $a_1 \neq a_2$, $b_1 \neq b_2$ and exclude the end-points of the interval $a \ominus b$ from the set $\{a_i - b_j\}_{i,j=1,2}$. The remaining two points determine a shorter interval, which we shall further denote by $a - b$. This is the non-standard subtraction studied in some detail in Section 1. (The non-standard division can be roughly described in a similar way; a detailed study of it is given in section 6.)

In order to illustrate the use of the non-standard subtraction, consider the inteval $c = \{\alpha - \alpha^2 \mid \alpha \in a\}$, where a is a given interval, such that $a \bigcap (0, 1/2) = \emptyset$. In this case we may write

$$c = \begin{cases} a - a^2, & \text{if } \ a \geq 1/2, \\ a \ominus a^2, & \text{if } \ a \leq 0. \end{cases}$$

The arithmetic operations can be introduced in various order. Relying on the fact that the reader is familiar with the standard arithmetic operations $+, \times, -, /$, we may define $b - a$ as the interval c, such that either $a + c = b$ or $b \ominus c = a$; we may introduce b/a as the interval d, such that either $ad = b$ or $b \oslash d = a$, etc.

We shall give the definitions of the arithmetic operations in a more natural order. We shall first introduce the basic arithmetic operations $+, \times, -, /$. Then we shall take advantage of the fact that the auxiliary operations are simple compositions of the basic ones, and namely: $a \oplus b = a - (-b)$, $a \ominus b = a + (-b)$, $a \otimes b = a/(1/b)$, $a \oslash b = a(1/b)$. Of course there is no need of special notations for the auxiliary operations (recall that there is no need to use a special notation for the operation $a + (-b)$ in the familiar interval arithmetic, either!); and we did not use such notations in our previous communications [5, 8]. However, many relations look much more simple and become easier to be memorized when using special notations for the auxiliary operations.

For convenience of the reader we shall confine our attention first to the non-standard subtraction of intervals, together with the standard addition and scalar multiplication. The non-standard division is considered in Section 6.

2 Non-standard (Basic) Subtraction of Intervals

Throughout the paper the compact intervals on the real line $\mathbb{R}$ are denoted by $a = [a_1, a_2]$, $b = [b_1, b_2]$, $c = [c_1, c_2]$,..., and the reals by $\alpha, \beta, \gamma, \ldots$. In the set $I(\mathbb{R})$ of compact intervals on $\mathbb{R}$ we define the following three operations:

i) basic arithmetic addition:

$$[a_1, a_2] + [b_1, b_2] := [a_1 + b_1, a_2 + b_2];$$

ii) scalar multiplication over $\mathbb{R}$:

$$\alpha \circ [a_1, a_2] := [\min\{\alpha a_1, \alpha a_2\}, \max\{\alpha a_1, \alpha a_2\}];$$

iii) basic arithmetic subtraction:

$$[a_1, a_2] - [b_1, b_2] := [\min\{a_1 - b_1, a_2 - b_2\}, \max\{a_1 - b_1, a_2 - b_2\}].$$

The product $(-1) \circ a$ is briefly denoted by $-a$; the interval $[\alpha, \alpha]$ is also written as α. The interval $a + (-b)$ will be denoted by $a \ominus b$. This familiar operation will be further called auxiliary subtraction; we have $a \ominus b = [a_1 - b_2, a_2 - b_1]$. The interval $a - (-b)$ will be denoted by $a \oplus b$. The operation $\oplus$ will be called auxiliary addition; we have $a \oplus b = [\min\{a_1 + b_2, a_2 + b_1\}, \max\{a_1 + b_2, a_2 + b_1\}]$.

Consider the set $I(\mathbb{R})$ together with the operations $+$, $\circ$ and $-$. The following relations (some of which are well-known) hold true in the space $< I(\mathbb{R}), +, \circ, - >$:

(R1) $a + b = b + a$ and $(a + b) + c = a + (b + c)$;

(R2) $\alpha(b + c) = \alpha b + \alpha c$;

(R3) $\alpha(b - c) = \alpha b - \alpha c$;

(R4) $(\alpha + \beta)c = \begin{cases} \alpha c + \beta c, & \text{if } \alpha\beta \geq 0, \\ \alpha c \oplus \beta c, & \text{if } \alpha\beta < 0; \end{cases}$

(R5) $\alpha(\beta c) = (\alpha\beta)c$;

(R6) $1 \circ a = a$;

(R7) $0 \circ a = 0$;

(R8) $a \oplus b = b \oplus a$.

We pause here to derive some direct corollaries from these relations. We note that $a + 0 = 0 + a = a$, $a - 0 = a$, $0 - a = -a$ and $a - a = 0$ for every interval a. Relation (R4) can also be written:

$$(\alpha - \beta)c = \begin{cases} \alpha c - \beta c, & \text{if } \alpha\beta \geq 0, \\ \alpha c \ominus \beta c, & \text{if } \alpha\beta < 0. \end{cases}$$

Relation (R8) can be written $(-a) - b = (-b) - a$ or $a \ominus b = (-b) + a$. (R8) and (R3) imply $a - b = (-b) - (-a) = -(b - a)$.

We shall formulate now four more relations by means of the width function $w : I(\mathbb{R}) \longrightarrow \mathbb{R}$, defined by $w([a_1, a_2]) = a_2 - a_1$. By the way, w satisfies the following relations:

(W1) $w(a) \geq 0$;

(W2) $w(\alpha \circ a) = |\alpha| w(a)$;

(W3) $w(a + b) = w(a) + w(b)$;

(W3) $w(a - b) = |w(a) - w(b)|$.

By means of w the basic subtraction can be written:

$$a - b = \begin{cases} [a_1 - b_1, a_2 - b_2], & \text{if } \ w(a) \geq w(b), \\ [a_2 - b_2, a_1 - b_1], & \text{if } \ w(a) < w(b), \end{cases}$$

showing that the basic subtraction summarizes in a common formula the so-called Hukuhara-differences. The first such difference is used for instance, in [12, 14]; both differences are studied in [2].

Similarly we can write

$$a \oplus b = \begin{cases} [a_1 + b_2, a_2 + b_1], & \text{if } \ w(a) \geq w(b), \\ [a_2 + b_1, a_1 + b_2], & \text{if } \ w(a) < w(b). \end{cases}$$

We recall that the operation $a \oplus b$ for $w(a) \geq w(b)$ (called *pseudo-addition*) is used in [13].

According to (R8) the auxiliary addition is commutative. It is also semiassociative in the sense that:

(R9) $(a \oplus b) \oplus c = \begin{cases} a \oplus (b \oplus c), & \text{if } \ w(b) \geq \max\{w(a), w(c)\}, \\ b \oplus (c \oplus a), & \text{if } \ w(a) \geq \max\{w(b), w(c)\}. \end{cases}$

In other words the widest interval should be brackets in both sides of the equality. For instance, if c is the widest interval, then one can write $a \oplus (b \oplus c) = (a \oplus c) \oplus b$, etc.

In the next three relations we shall make use of the following (logical) propositions with respect to the intervals a, b, c, d:

$$\begin{aligned} \mathbb{B}_1 &= (w(a) \leq w(c) \text{ .and. } w(b) \leq w(d)) \text{ .or. } (w(a) \geq w(c) \text{ .and. } w(b) \geq w(d)), \\ \mathbb{B}_2 &= (w(a) \geq w(c) \text{ .and. } w(b) \leq w(d)) \text{ .or. } (w(a) \leq w(c) \text{ .and. } w(b) \geq w(d)), \\ \mathbb{D}_1 &= (w(a) \leq w(b) \text{ .and. } w(c) \leq w(d)) \text{ .or. } (w(a) \geq w(b) \text{ .and. } w(c) \geq w(d)), \\ \mathbb{D}_2 &= (w(a) \geq w(b) \text{ .and. } w(c) \leq w(d)) \text{ .or. } (w(a) \leq w(b) \text{ .and. } w(c) \geq w(d)). \end{aligned}$$

Remark: Propositions $\mathbb{B}_1$–$\mathbb{D}_2$ are written in a form suitable to be used in section 3, where general intervals are considered. For intervals from $I(\mathbb{R})$ we can simply write $(w(a) - w(c))(w(b) - w(d)) \geq 0$ instead of $\mathbb{B}_1$ etc.

We can formulate now the following relations:

(R10) $(a + b) - (c + d) = \begin{cases} (a - c) + (b - d), & \text{if } \ \mathbb{B}_1, \\ (a - c) \oplus (b - d), & \text{if } \ \mathbb{B}_2; \end{cases}$

(R11) $(a - b) + (c - d) = \begin{cases} (a + c) - (b + d), & \text{if } \ \mathbb{D}_1, \\ (a \oplus c) - (b \oplus d), & \text{if } \ \mathbb{D}_2 \text{ and } B_1, \\ (a \oplus c) \ominus (b \oplus d), & \text{if } \ \mathbb{D}_2 \text{ and } B_2; \end{cases}$

(R12) $(a-b)-(c-d) = \begin{cases} (a-c)-(b-d), & \text{if } \mathbb{D}_1 \text{ and } B_1, \\ (a-c)\ominus(b-d), & \text{if } \mathbb{D}_1 \text{ and } B_2, \\ (a\ominus c)-(b\ominus d), & \text{if } \mathbb{D}_2. \end{cases}$

Detailed verifications of the above relations and some corollaries are given in [8]. In the same paper the problem of a basis in $I(\mathbb{R})$ is also discussed.

We shall say that a function $||\cdot|| : I(\mathbb{R}) \longrightarrow [0,\infty)$ is a norm in $I(\mathbb{R})$ if it satisfies the relations:

(N1) $||a|| > 0$, for $a \neq 0$; $||0|| = 0$;
(N2) $||\alpha a|| = |\alpha| ||a||$;
(N3) $||a+b|| = ||a|| + ||b||$;
(N4) $||a-b|| = ||a|| + ||b||$.

It is readlily verified that $||[a_1,a_2]|| = \max\{|a_1|,|a_2|\}$ is a norm in $I(\mathbb{R})$; we shall also denote this norm by $||\cdot||_{I(\mathbb{R})}$. The norm $||\cdot||_{I(\mathbb{R})}$ generates the distance $r(a,b) = \max\{|a_1-b_1|,|a_2-b_2|\}$, that is we have $||a-b|| = r(a,b)$.

More generally, one can consider abstract interval space with basic subtraction. In such spaces one can introduce a norm; and then study normed interval spaces.

3 Normed Interval Spaces

Denote by L a normed lattice, that is a vector lattice with a monotone norm $||\cdot||_L$ [4], p. 376. (Monotonicity of $||\cdot||$ means that $|a_1| \leq |a_2|$, $a_1, a_2 \in L$, implies $||a_1|| \leq ||a_2||$; where $|a_i| = \sup(a_i, 0) + \sup(-a_i, 0) = \sup(a_i, -a_i)$.)

The set $[a_1,a_2] := \{\alpha \in L \mid a_1 \leq \alpha \leq a_2\}$, where $a_1 \leq a_2$, $a_1, a_2 \in L$, is called an *interval* in L. The set of all intervals in L is denoted by $I(L)$.

Define the following operations in $I(L)$:

i) addition:

$$[a_1,a_2]+[b_1,b_2] := [\inf(a_1+b_1,a_2+b_2), \sup(a_1+b_1,a_2+b_2)] = [a_1+b_1,a_2+b_2];$$

ii) subtraction:

$$[a_1,a_2]-[b_1,b_2] := [\inf(a_1-b_1,a_2-b_2), \sup(a_1-b_1,a_2-b_2)];$$

iii) scalar multiplication over $\mathbb{R}$:

$$\alpha[a_1,a_2] := [\inf(\alpha a_1, \alpha a_2), \sup(\alpha a_1, \alpha a_2)] = \begin{cases} [\alpha a_1, \alpha a_2], & \text{if } \alpha \geq 0, \\ [\alpha a_2, \alpha a_1], & \text{if } \alpha < 0. \end{cases}$$

We shall call the set $I(L)$ together with these three operations an *interval space over* L.

Consider the width function $w : I(L) \longrightarrow L$ defined by $w([a_1,a_2]) = a_2 - a_1$. The function w satisfies relations (W1)–(W4).

It is easily verified that relations (R1)–(R12) hold true in an arbitrary interval space over a vector lattice (by the remark that in relation (R9) "max" should be replaced by "sup").

An interval space $I(L)$ over L, where L is a normed lattice can be normed by means of $||[a_1, a_2]||_{I(L)} := \max\{||a_1||_L, ||a_2||_L\}$. It is easlily seen that $||\cdot||_{I(L)}$ satisfies relations (N1)–(N4).

Therefore we may call an interval space over a normed lattice a *normed interval space*.

Convergence in norm in a normed interval space is defined as usually as follows: the sequence $\{x^{(n)}\}_{n=1}^{\infty}$ in $I(L)$ is said to converge in norm to $x \in I(L)$, if $\lim_{n\to\infty} ||x^{(n)} - x||_{I(L)} = 0$. It is easily seen that a normed interval space $I(L)$ is a complete space in the metric $r(x,y) = ||x-y||_{I(L)}$, if L is complete (that is L is a Banach lattice [3], p. 366; [4], p. 376).

4 Interval Operators

Consider two normed lattice L_1 and L_2 and the corresponding normed interval spaces $I(L_1)$ and $I(L_2)$. An interval operator is a mapping $U : D \longrightarrow I(L_2)$, where D is a subset of $I(L_1)$. U is continuous at $f \in D$, if $f^{(n)} \in D$, $f^{(n)} \longrightarrow f$ (in the sense of $||\cdot||_{I(L_1)}$) implies $U(f^{(n)}) \longrightarrow U(f)$ (in the sense of $||\cdot||_{I(L_2)}$).

As an example of an interval operator consider the space C of all continuous (single-valued) functions in $[0,1]$ with the uniform norm $||\varphi||_C = \max_{0\le x\le 1} |\varphi(x)|$ and the corresponding interval space $I(C)$.

The space $I(C)$ is normed by

$$||f||_{I(C)} = ||[f_1, f_2]||_{I(C)} = \max\{||f_1||_C, ||f_2||_C\} = \max_{0\le x\le 1} ||f(x)||_{I(R)}.$$

Consider the operator $B_n : I(C) \longrightarrow I(C)$ defined by

$$B_n(f;x) = \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k}.$$

The operator $B_n(f)$ is quasilinear in the sense that $B_n(\alpha f) = \alpha B_n(f)$, $B_n(f_1 + f_2) = B_n(f_1) + B_n(f_2)$, and is monotone: $B_n(f) \ge 0$, for $f \ge 0$. We may note that the equality $B_n(f_1 - f_2) = B_n(f_1) - B_n(f_2)$ is not satisfied in general, but when $w(f_1) - w(f_2) = \text{const}$.

The following proposition holds true:

Proposition 1 *For every $\varepsilon > 0$, there exists a positive integer n such that $||f(x) - B_n(f;x)||_{I(C)} < \varepsilon$.*

5 Differentiation and Integration of Interval Functions

An interval function (of a real variable) is a mapping $f : D \longrightarrow I(\mathbb{R})$, where D is a fixed subset of $\mathbb{R}$. Continuity and limits of f are understood in the sense of the norm $||f||_{I(\mathbb{R})}$, which leads to the same concepts as in [10].

We shall say that an interval function is w-increasing in D, if $w(f(x))$ is increasing in D.

The derivative of the interval function f at $x \in D$ is:

$$f'(x) := \lim_{h \to 0} (f(x+h) - f(x))/h,$$

if the limit exists.

Several propositions using this definition are formulated below. Let us] remark that our definition is very close (but not the same) to the definitions proposed in [13, 14].

Proposition 2 (Mean-value theorem for interval functions) *If the interval function f is continuous in $\Delta = [\alpha, \beta]$ and differentiable in (α, β), then $f(\beta) - f(\alpha) \subset f'(\Delta)(\beta - \alpha)$, where $f'(\Delta) = \bigcup_{\delta \in \Delta} f'(\delta)$.*

The content of this theorem is very close to the corresponding theorem in [13]. Here is a simple corollary of the mean-value theorem:

Proposition 3 *If the interval function f is continuous over $[\alpha, \beta]$ and $f'(x) = 0$ for $\alpha < x < \beta$, then f is a constant, that is $f \equiv c$, $c \in I(\mathbb{R})$.*

The integral of a function $f : [\alpha, \beta] \longrightarrow I(\mathbb{R})$ is defined after [1, 11] by

$$\int_\alpha^\beta f(x)dx = \left[\int_\alpha^\beta f_1(x)dx, \int_\alpha^\beta f_2(x)dx \right],$$

where the functions f_i, defined by $f = [f_1, f_2]$, are integrable.

Proposition 4 *If the interval function f is continuous in $[\alpha, \beta]$ and $\alpha < x < \beta$, then the interval function F, defined by means of $F(x) = \int_\alpha^x f(t)dt$ is differentiable in $[\alpha, \beta]$ and $F' = f$.*

Proposition 5 *If the interval function F is w-increasing in $[\alpha, \beta]$ and possesses a continuous derivative $F' = f$ in $[\alpha, \beta]$, then*

$$\int_\alpha^\beta f(x)dx = F(\beta) - F(\alpha).$$

The last two propositions can be generalized by considering "absolute continuous interval functions" [9].

Remark. An interesting theory can be developed [7] when using a more general definition of derivative, based on the concept of S-limit [15]. In [7] we give a comparison between the derivative $\mathrm{Slim}_{h \to 0}(f(x+h) - f(x))/h$ and the derivative $\mathrm{Slim}_{h \to 0}(f(x+h) \ominus f(x))/h$, proposed by Bl. Sendov [16].

6 The Basic Division of Intervals

In what follows we shall mainly consider intervals on $\mathbb{R}$, which do not contain zero. The set of all such intervals will be denoted by $I^\star(\mathbb{R})$. We shall introduce some further notations for intervals from $I^\star(\mathbb{R})$. An interval $x \in I^\star(\mathbb{R})$ is either positive ($x > 0$) or negative ($x < 0$), and we shall write $s(x) = 1$, if $x > 0$, and $s(x) = -1$, if $x < 0$. We denote the end-point of $x \in I^\star(\mathbb{R})$, which is closer to zero by x_c, and the other end-point by x_d; thus we have $|x_c| \leq |x_d|$ and

$$x = \begin{cases} [x_c, x_d], & \text{if } \ x > 0, \\ [x_d, x_c], & \text{if } \ x < 0. \end{cases}$$

Consider the function $v(x)$ defined in $I(\mathbb{R})$ by $v(x) = x_d/x_c$. We have: $v(x) \leq 1$. The function v plays an important role further on.

We shall recall now the basic multiplication $ab = \{\alpha\beta \mid \alpha \in a, \ \beta \in b\}$, $a, b \in I(\mathbb{R})$, expressing it by means of the end-points of the intervals. If one of the intervals, say b, does not contain zero, we may write:

$$ab = \begin{cases} [a_c b_c, a_d b_d], & \text{if } \ a \not\ni 0, \ \ s(a) = s(b), \\ [a_d b_d, a_c b_c], & \text{if } \ a \not\ni 0, \ \ s(a) \neq s(b), \\ (b_d)a, & \text{if } \ a \ni 0, \end{cases}$$

for all $a \in I(\mathbb{R})$, $b \in I^\star(\mathbb{R})$. Using Sunaga's notation $\alpha \bigvee \beta := [\min\{\alpha, \beta\}, \max\{\alpha, \beta\}]$, $\alpha, \beta \in \mathbb{R}$ [17], the last formula can be shortly written:

$$ab = \begin{cases} (a_c b_c) \bigvee (a_d b_d), & \text{if } \ a \not\ni 0, \\ (b_d)a, & \text{if } \ a \ni 0. \end{cases}$$

We define the basic division by means of a similar formula:

$$a/b := \begin{cases} (a_c/b_c) \bigvee (a_d/b_d), & \text{if } \ a \not\ni 0, \\ (1/b_d)a, & \text{if } \ a \ni 0, \end{cases}$$

for all $a \in I(\mathbb{R})$, $b \in I^\star(\mathbb{R})$. Of course we can define a/b without using "$\bigvee$", for instance:

$$a/b = \begin{cases} [a_c/b_c, a_d/b_d], & \text{if } \ a \not\ni 0, \ \ s(ab)(v(a) - v(b)) \geq 0, \\ [a_d/b_d, a_c/b_c], & \text{if } \ a \not\ni 0, \ \ s(ab)(v(a) - v(b)) < 0, \\ (1/b_d)a, & \text{if } \ a \ni 0. \end{cases}$$

We note that $1/b = 1/[b_1, b_2] = [b_2^{-1}, b_1^{-1}]$. Therefore $a(1/b)$ is the standard arithmetic division, producing generally a wider interval than a/b. We shall further call the familiar operation $a(1/b)$ *auxiliary division* and shall denote it by $a \oslash b = a(1/b)$. Similarly, we define the *auxiliary multiplication* by $a \otimes b := a/(1/b)$.

The basic division has the important property that $a/a = 1$ for $a \not\ni 0$. The properties of the basic division can be easily obtained by the observation that the algebraic structure of the space $< I(\mathbb{R}), \times, / >$ is analogous to the algebraic structure of the space $< I(\mathbb{R}), +, - >$ as it is shown in [8]. The function v plays thereby the role of the function w in the space $< I(\mathbb{R}), +, - >$.

The auxiliary multiplication produces generally a narrower interval than the basic multiplication. It is a commutative operation, $a \otimes b = b \otimes a$. It is also a semiassociative operation, that is

$$(a \otimes b) \otimes c = \begin{cases} (b \otimes c) \otimes a, & \text{if } \ v(b) \geq \max\{v(a), v(c)\}, \\ (a \otimes c) \otimes b, & \text{if } \ v(a) \geq \max\{v(b), v(c)\}. \end{cases}$$

In what follows we shall use all arithmetic operations. The following table summarizes the whole set of operations used in extended interval arithmetic

operations in extended interval arithmetic	basic operations	auxiliary operations
standard operations	$a + b$ $a \times b$	$a \ominus b = a + (-b)$ $a \oslash b = a \times (1/b)$
non-standard operations	$a - b$ a/b	$a \oplus b = a - (-b)$ $a \otimes b = a/(1/b)$

We shall confine first our attention to the distributive laws in the extended interval arithmetic, that is we shall formulate some relations between the operations $+, -$ at one side and the operations $\times, /$ at the other side.

7 Distributive Laws in Extended Interval Arithmetic

As it is well-known, $a, b, a+b \in I^\star(\mathbb{R})$ and $ab > 0$, then $(a+b)c = ac + bc$. Using extended interval arithmetic we can give the following solution to the case $ab < 0$.

Proposition 6 *If $a, b, c, a+b \in I^\star(\mathbb{R})$ and $ab < 0$, then*

$$(a+b)c = \begin{cases} a \otimes c + bc, & \text{if } \ v(c) \leq v(a), \ \ s(b(a+b)) \leq 0, \\ ac + b \otimes c, & \text{if } \ v(c) \leq v(a), \ \ s(b(a+b)) < 0, \\ a \otimes c \oplus bc, & \text{if } \ v(c) > v(a), \ \ s(b(a+b)) \leq 0, \\ ac \oplus b \otimes c, & \text{if } \ v(c) > v(a), \ \ s(b(a+b)) < 0. \end{cases}$$

Similarly, for $(a-b)c$ we can state the following proposition:

Proposition 7 *If $a, b, c, a-b \in I^\star(\mathbb{R})$ and $ab > 0$, then*

$$(a-b)c = \begin{cases} ac - bc, & \text{if } \ s(a(a-b)) \geq 0, \\ a \otimes c - b \otimes c, & \text{if } \ s(a(a-b)) < 0, \ \ (v(a) - v(c))(v(b) - v(c)) \geq 0, \\ a \otimes c \ominus b \otimes c, & \text{if } \ s(a(a-b)) < 0, \ \ (v(a) - v(c))(v(b) - v(c)) < 0. \end{cases}$$

Proposition 8 *If $a, b, c, a-b \in I^\star(\mathbb{R})$ and $ab < 0$, then*

$$(a-b)c = \begin{cases} ac \ominus b \otimes c, & \textit{if } w(a) \geq w(b), \; v(c) \geq v(b), \\ ac - b \otimes c, & \textit{if } w(a) \geq w(b), \; v(c) < v(b), \\ a \otimes c \ominus bc, & \textit{if } w(a) < w(b), \; v(c) \geq v(a), \\ a \otimes c - bc, & \textit{if } w(a) < w(b), \; v(c) < v(a). \end{cases}$$

A paper by N. Dimitrova, containing more similar propositions is now in preparation.

Here are two more examples, illustrating the possibilities of the extended interval arithmetic.

Proposition 9 *If a, b are two intervals, such that $a-b$ and $a+b$ does not contain zero, then*

$$a^2 - b^2 = \begin{cases} (a-b)(a+b), & \textit{if } s(a(a-b))(w(a)-w(b)) \geq 0, \\ (a-b) \otimes (a+b), & \textit{if } s(a(a-b))(w(a)-w(b)) < 0, \end{cases}$$

where $a^2 := a \times a$.

Proposition 10 *For every $a, b \in I(\mathbb{R})$ we have $\exp(a+b) = \exp(a)\exp(b)$ and $\exp(a-b) = \exp(a)/\exp(b)$, where $\exp(a) = \exp([a_1, a_2]) := [\exp(a_1), exp(a_2)]$.*

8 Matrix Computations with Intervals

As it is well-known the set of solutions of a linear equation $\alpha\xi = \beta$, when $\alpha \in a$, $\beta \in b$ (and $a \in I^\star(\mathbb{R})$, $b \in I(\mathbb{R})$) can be easily expressed by means of (standard) interval arithmetic: $x = b \oslash a$.

As a next example consider the system

$$\begin{aligned} \alpha\xi + \eta &= -1, \\ \beta\xi + \eta &= 1, \end{aligned}$$

where $\alpha \in a$, $\beta \in b$ (and $a \ominus b \not\ni 0$). The sets $x, y \in I(\mathbb{R})$ of solutions for ξ and η can be again expressed by a, b, using standard interval arithmetic:

$$\begin{aligned} x &= (-2) \oslash (a \ominus b), \\ y &= 1 + 2 \oslash (a \oslash b \ominus 1). \end{aligned}$$

However, the standard interval arithmetic is of little help, when considering the more complicate case:

$$\begin{aligned} \alpha_1\xi + \eta &= \beta_1, \\ \alpha_2\xi + \eta &= \beta_2, \end{aligned}$$

where $\alpha_1 \in a_1$, $\alpha_2 \in a_2$, $\beta_1 \in b_1$, $\beta_2 \in b_2$. If $a_1 \ominus a_2 \not\ni 0$, we obtain for the set of solutions for ξ:

$$x = (b_1 \ominus b_2) \oslash (a_1 \ominus a_2).$$

A much more difficult task is to find an expression for $y = \{\eta\}$. A student of mine, I. Nedkov, obtained the following result:

$$y = \begin{cases} b_2 + (b_2 \ominus b_1) \oslash (a_1 \oslash a_2 - 1), & \text{if } \; a_2 \subset 0 \bigvee a_1, \\ b_2 \oplus (b_2 \ominus b_1) \oslash (a_1 \oslash a_2 - 1), & \text{if } \; a_1 \subset 0 \bigvee a_2, \\ b_2 + (b_2 - b_1) \oslash (a_1 \oslash a_2 - 1), & \text{if } \; 0 \in a_1 \bigvee a_2 \text{ and } w(b_2) \geq w(b_1), \\ b_2 + (b_2 - b_1)/(a_1 \oslash a_2 - 1), & \text{if } \; 0 \in a_1 \bigvee a_2,\; w(b_2) > w(b_1) \text{ and } A \geq 0, \\ b_2 \oplus (b_2 - b_1)/(a_1 \oslash a_2 - 1), & \text{if } \; 0 \in a_1 \bigvee a_2,\; w(b_2) > w(b_1) \text{ and } A < 0, \end{cases}$$

where

$$A = v(a_1 - a_2)^{\mathrm{sign}(w(a_1) - w(a_2))} - v(b_2 - b_1)/v(a_2).$$

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