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**A Complete Axiomatization of  
Kaucher Interval Arithmetic**

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# A Complete Axiomatization of Kaucher Interval Arithmetic

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## Abstract

Algebraic systems, abstracting properties of intervals, are discussed. Certain algebraic structures close to vector spaces, ordered rings, fields and algebras are axiomatically introduced and studied.

Key words: intervals, centred intervals, interval arithmetic, generalized interval arithmetic, quasivector space, e-ring, e-linear space, e-algebra.

## 1 Introduction

In numerical analysis one often works with approximate numbers and errors (error bounds). An *approximate number*  $x$  that is known up to an *error bound*  $\delta \geq 0$  represents any real number  $\tilde{x}$  between  $x - \delta$  and  $x + \delta$ , i. e.  $x - \delta \leq \tilde{x} \leq x + \delta$ . Alternatively we can write  $|x - \tilde{x}| \leq \delta$  or  $\tilde{x} \in x + [-\delta, \delta]$ . From this point of view an error bound  $\delta \geq 0$  can be identified with the *centred (origin symmetric, 0-symmetric) interval*  $[-\delta, \delta]$ . Thus formally the terms *centred interval* and *error (error bound)* can be used as synonyms. Going a step further, an interval  $A = [a' - a'', a' + a'']$  can be conceived as a pair  $A = (a'; a'')$  consisting of a real number  $a'$  and an error bound  $a'' \geq 0$ ; thereby  $a''$  is considered as the (upper bound of the) error in the (exact) number  $a'$ . Thus again the terms interval and approximate number can be considered as synonyms. Other commonly used notation for an approximate number  $A = (a'; a'')$  is  $A = a' \pm a''$ .

In many situations it has been useful and natural to introduce arithmetic operations and relations for approximate numbers (intervals) in the way we introduce operations and relations for real numbers. The resulting arithmetic system is known as interval arithmetic. There is a natural tendency in interval arithmetic to follow the development of real arithmetic: study of

algebraic properties, classification into known algebraic systems, axiomatization, etc. From one side,  $n$ -dimensional intervals are added and multiplied by scalars like real vectors, on the other side one-dimensional intervals are added, subtracted, multiplied and divided like real numbers. However, intervals do not form neither vector spaces nor rings. Therefore many authors have studied the algebraic properties of intervals and have tried to formulate algebraic systems abstracting their properties, see e.g. [4], [6], [8], [9], [21], [23], [29], [32], [33], [36], [38] et al. However, still interval arithmetic has not been completely axiomatized. The present work intends to fill up this gap by introducing certain new algebraic systems emanating from the algebraic properties of intervals.

Interval arithmetic has specific peculiarities in comparison to real arithmetic. For example, multiplying an error (centred interval) by  $-1$  does not influence the result. Another peculiarity is that the ordering of errors has the meaning of inclusion (e. g.  $\delta_1 \leq \delta_2$  means  $[-\delta_1, \delta_1] \subseteq [-\delta_2, \delta_2]$ ), while the ordering of real numbers is preceding " $\leq$ " (less than or equal to). Thus for approximate numbers (at least) two different orderings appear, one of which having the meaning of preceding, and another one having the meaning of inclusion. Inclusion isotonicity is an important practically useful property of interval arithmetic, which is not valid in real arithmetic, e. g. multiplication by scalars is (inclusion) order isotone in interval arithmetic, but is not (preceding) order isotone in real arithmetic ( $a \leq b \implies \gamma a \leq \gamma b$  not true in general).

Since any interval  $A = (a'; a'')$  is a sum of a real number  $a'$  and a centred interval  $(0; a'') = [-a'', +a'']$  one should investigate the algebraic properties of centred intervals as long as the algebra of real numbers is well-known. Regarding addition of errors (centred intervals) the situation is the same as with real numbers. Being identified with nonnegative numbers, errors form — as nonnegative numbers do — an additive commutative monoid. Hence they can be naturally embedded in a group (as we do with numbers); thereby some new—improper—elements are introduced. Then appropriate definitions of inclusion and multiplication are needed for the new elements. We note that inclusion and multiplication can be extended in the additive group in such a way that multiplication is inclusion isotone. This practically important extension of multiplication has been thoroughly studied in [6], [7], [8]. In recognition to E. Kaucher's contribution this extension of interval arithmetic is known as Kaucher interval arithmetic.

This work continues Kaucher's studies on the fundamentals of interval arithmetic by developing an axiomatic algebraic theory of errors (centred

intervals over  $\mathbb{R}$ ) and approximate numbers (intervals over  $\mathbb{R}$ ). We have shown in [12], [13], [16] that centred intervals play a special role in the algebraic study of intervals by investigating in some detail properties related to multiplication by scalars. Here we briefly outline this approach and continue our study by developing the abstract system related to (inner) multiplication — an operation that has the property of being (unlike multiplication of reals) order isotone. In the short note [17] we sketched the axiomatization of the arithmetic for centred intervals. Here we develop the axiomatization of generalized interval arithmetic.

In Section 2 we begin by introducing certain operations and systems for real numbers that are of interest to interval arithmetic; then we outline the latter in a manner suitable for our purposes. In Section 3 we recall the theory of quasivector spaces of centred intervals, and introduce order obtaining so-called e-linear spaces; we then introduce inner multiplication and define e-rings and e-fields. In Section 4 an e-algebra as a combination between an e-ring and an e-linear space is introduced. We then consider the direct sum of a ring and an e-algebra and define multiplication, obtaining thus an interval algebra. It has been shown that the space of intervals over  $\mathbb{R}$  is a special case of such an interval algebra.

## 2 Preliminaries: Real and Interval Arithmetic

We shall first point out several issues from real arithmetic related to interval arithmetic, then we shall briefly present the basics of interval arithmetic in a form suitable for our study.

### 2.1 Real Arithmetic: Issues Related to Intervals

**The ordered field of reals.** By  $\mathbb{R}$  we denote the set of reals;  $\mathbb{R}_D = (\mathbb{R}, +, \cdot, \leq)$  is the linearly ordered (l. o.) field of reals (the reason why we use distinct notations  $\mathbb{R}$ ,  $\mathbb{R}_D$  for the set, resp. the field of reals will become clear in the sequel). As usually  $\mathbb{R}^n$  denotes the Cartesian product  $\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}$  ( $n$  times). For  $a = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^n$ ,  $b = (\beta_1, \dots, \beta_k) \in \mathbb{R}^n$ , the partial order “ $\leq$ ” is given component-wise by  $a \leq b \iff a_i \leq b_i$  for all  $i = 1, \dots, n$ . The  $n$ -dimensional (partially) ordered real vector space is denoted  $\mathbb{R}_D^n = (\mathbb{R}^n, +, \mathbb{R}_D, \cdot, \leq)$ ,  $n \geq 1$ ; here “ $+$ ” denotes vector addition and “ $\cdot$ ” denotes multiplication by scalars:

$$(\alpha_1, \alpha_2, \dots, \alpha_k) + (\beta_1, \beta_2, \dots, \beta_k) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_k + \beta_k), \quad (1)$$

$$\gamma \cdot (\alpha_1, \dots, \alpha_k) = \gamma(\alpha_1, \dots, \alpha_k) = (\gamma\alpha_1, \dots, \gamma\alpha_k), \quad \gamma \in \mathbb{R}; \quad (2)$$

again we use different notation for the set  $\mathbb{R}^n$  and the (ordered) vector space  $\mathbb{R}_D^n$ .

### 2.1.1 The quasivector multiplication by real scalars

Consider the system  $(\mathbb{R}^k, +, \mathbb{R}_D, *, \leq)$ , consisting of the set  $\mathbb{R}^k$  of all  $k$ -tuples  $a = (\alpha_1, \alpha_2, \dots, \alpha_k)$ ,  $\alpha_i \in \mathbb{R}$ ,  $i = 1, \dots, k$ , together with the operations (1) and

$$\gamma * a = \gamma * (\alpha_1, \dots, \alpha_k) = |\gamma|(\alpha_1, \dots, \alpha_k) = (|\gamma|\alpha_1, \dots, |\gamma|\alpha_k), \quad \gamma \in \mathbb{R}. \quad (3)$$

The operation (3) is called *quasivector multiplication by scalars*; the latter differs from familiar vector (linear) multiplication by scalars (2) as far as negative scalars are concerned. For example, the quasivector product of the triple  $(1, 1, 1)$  by the scalar  $-2$  is  $-2 * (1, 1, 1) = (2, 2, 2)$ , whereas the vector product of  $(1, 1, 1)$  by  $-2$  yields  $-2 \cdot (1, 1, 1) = (-2, -2, -2)$ . For a better distinction we shall sometimes use the adjective “linear” for the vector multiplication by scalars.

The system  $\mathbb{R}_{ED}^k = (\mathbb{R}^k, +, \mathbb{R}_D, *, \leq)$  will be further called *the canonical  $k$ -dimensional quasivector space*. Note that multiplication by  $-1$  (negation) in  $(\mathbb{R}^k, +, \mathbb{R}_D, *)$  is same as identity; however there is an opposite operator:  $\text{opp}(\alpha_1, \alpha_2, \dots, \alpha_k) = (-\alpha_1, -\alpha_2, \dots, -\alpha_k)$ .

The system  $\mathbb{R}_{ED}^k = (\mathbb{R}^k, +, \mathbb{R}_D, *, \leq)$  differs from the vector space  $\mathbb{R}_D^k = (\mathbb{R}^k, +, \mathbb{R}_D, \cdot)$  by having a *quasivector* multiplication by scalars instead of a linear one. In the space  $\mathbb{R}_{ED}^k = (\mathbb{R}^k, +, \mathbb{R}_D, *, \leq)$  the second distributive law  $(\alpha + \beta) * c = \alpha * c + \beta * c$  does not hold in general. Indeed, for  $\alpha = 1$ ,  $\beta = -1$  we have  $0 = (1 - 1) * c \neq 1 * c + (-1) * c = 2 * c$ . However, a so-called quasidistributive law takes place. To formulate it, we shall make use of the binary set  $\Lambda = \{+, -\}$  and the function  $\sigma : \mathbb{R}_D \rightarrow \Lambda$  defined by:

$$\sigma(\gamma) = \begin{cases} +, & \text{if } \gamma \geq 0, \\ -, & \text{if } \gamma < 0. \end{cases} \quad (4)$$

For  $a \in \mathbb{R}^k$  we make the convention  $a_+ = a$ ,  $a_- = -a$ . Then the symbolic notation  $a_\lambda$  for  $a \in \mathbb{R}^k$ ,  $\lambda \in \Lambda$ , makes sense; namely  $a_\lambda$  is either  $a = a_+$  or  $a_-$  according to the binary value of  $\lambda$ . Multiplication by scalars (3) is distributive generally over nonnegative scalars:  $(\alpha + \beta) * c = \alpha * c + \beta * c$ ,  $\alpha \geq 0, \beta \geq 0, c \in \mathbb{R}^k$ . It turns out that the latter is equivalent to the following *quasidistributive rule* treating all remaining cases for the signs of the scalars: for  $\alpha, \beta \in \mathbb{R}$  and  $c \in \mathbb{R}^k$

$$(\alpha + \beta) * c_{\sigma(\alpha+\beta)} = \alpha * c_{\sigma(\alpha)} + \beta * c_{\sigma(\beta)}. \quad (5)$$

Note that the rest of the vector-space laws for multiplication by scalars are still valid:  $\gamma * (a + b) = \gamma * a + \gamma * b$ ,  $\alpha * (\beta * c) = (\alpha\beta) * c$ ,  $1 * a = a$  for  $a, b, c \in \mathbb{R}^k$  and  $\alpha, \beta, \gamma \in \mathbb{R}$ .

Why we are interested in the quasivector multiplication by scalars (3)? The quasivector multiplication by scalars is order isotone, that is:  $a \leq b \implies \gamma * a \leq \gamma * b$ . This property is of considerable practical importance in interval arithmetic.

In a canonical quasivector space  $\mathbb{R}_{ED}^k = (\mathbb{R}^k, +, \mathbb{R}_D, *, \leq)$  one can readily introduce linear multiplication by scalar. Indeed, define in  $(\mathbb{R}^k, +, \mathbb{R}_D, *, \leq)$  the operation  $\star : \mathbb{R} \times \mathbb{R}^k \longrightarrow \mathbb{R}^k$ :

$$\gamma \star (\alpha_1, \dots, \alpha_k) = \begin{cases} \gamma * (\alpha_1, \dots, \alpha_k), & \text{if } \alpha \geq 0, \\ \gamma * (-\alpha_1, \dots, -\alpha_k), & \text{if } \alpha < 0. \end{cases} \quad (6)$$

We see that the operation “ $\star$ ” is derived from “ $*$ ” and opposite and is inherent for the system. It is easy to see that “ $\star$ ” is in fact the linear multiplication by scalars “ $\cdot$ ”. Indeed, for  $\gamma \geq 0$  looking at (6) and (3) we have  $\gamma \star (\alpha_1, \dots, \alpha_k) = \gamma * (\alpha_1, \dots, \alpha_k) = \gamma \cdot (\alpha_1, \dots, \alpha_k)$ . For  $\gamma < 0$  we have, again from (6), (3):

$$\begin{aligned} \gamma \star (\alpha_1, \dots, \alpha_k) &= \gamma * (-\alpha_1, \dots, -\alpha_k) = |\gamma| (-\alpha_1, \dots, -\alpha_k) \\ &= -\gamma (-\alpha_1, \dots, -\alpha_k) = \gamma \cdot (\alpha_1, \dots, \alpha_k), \end{aligned}$$

showing that “ $\star$ ” defined by (6) is the familiar linear multiplication by scalar.

This allows us to look at the system  $\mathbb{R}_{ED}^k = (\mathbb{R}^k, +, \mathbb{R}_D, *, \leq)$  as at the familiar vector space  $(\mathbb{R}^k, +, \mathbb{R}_D, \cdot, \leq)$  possessing one more—quasivector—multiplication by scalars:  $\mathbb{R}_{ED}^k = (\mathbb{R}^k, +, \mathbb{R}_D, \cdot, *, \leq)$ . It makes sense to use both multiplications “ $\cdot$ ”, “ $*$ ” in the same expression, e. g.  $ab + b * c$ .

Consider the direct sum  $\mathbb{R}_D^k \oplus \mathbb{R}_{ED}^k$ . The elements of  $\mathbb{R}_D^k \oplus \mathbb{R}_{ED}^k$  are denoted  $a = (a'; a'')$ , wherein  $a', a'' \in \mathbb{R}^k$ .

*Addition of  $a, b \in \mathbb{R}_D^k \oplus \mathbb{R}_{ED}^k$  is:*

$$\begin{aligned} a + b &= (a'; a'') + (b'; b'') = (a' + b'; a'' + b'') \\ &= (a'_1 + b'_1, \dots, a'_n + b'_n; a''_1 + b''_1, \dots, a''_n + b''_n). \end{aligned} \quad (7)$$

*Multiplication of an element of  $\mathbb{R}_D^k \oplus \mathbb{R}_{ED}^k$  by real scalars is*

$$\begin{aligned} \gamma * a &= \gamma * (a'; a'') = (\gamma a'; |\gamma| a'') \\ &= (\gamma a'_1, \dots, \gamma a'_n; |\gamma| a''_1, \dots, |\gamma| a''_n). \end{aligned} \quad (8)$$

We thus arrive to the system  $(\mathbb{R}_D^k \oplus \mathbb{R}_{ED}^k, +, \mathbb{R}_D, *)$ .

### 2.1.2 The e-multiplication of reals

Consider the following operation for  $a, b \in \mathbb{R}_D$ :

$$a \times b = \begin{cases} ab, & \text{if } a \geq 0, b \geq 0, \\ -ab, & \text{if } a \leq 0, b < 0, \\ 0, & \text{if } a > 0, b < 0 \text{ or } a < 0, b > 0. \end{cases} \quad (9)$$

For example:  $2 \times 3 = 6$ ,  $(-2) \times (-3) = -6$ ,  $-2 \times 3 = 0$ . To distinguish between the two operations for multiplication we shall call the familiar operation “ $\cdot$ ” *linear multiplication*, and “ $\times$ ” — *e-multiplication*. The set  $\mathbb{R}$  is closed with respect to (9). Hence we can consider the system  $\mathbb{R}_E = (\mathbb{R}, +, \times, \leq)$ , having the operation e-multiplication “ $\times$ ” instead of the linear multiplication “ $\cdot$ ” together with the same operation “ $+$ ” and the same order relation “ $\leq$ ” as in  $\mathbb{R}_D$ .

The e-multiplication (9) is associative:

$$(a \times b) \times c = a \times (b \times c) = \begin{cases} abc, & \text{if } \sigma(a) = \sigma(b) = \sigma(c), \\ 0, & \text{otherwise .} \end{cases} \quad (10)$$

It is also commutative:  $a \times b = b \times a$ . In addition the e-multiplication (9) is order isotone:  $a \leq b \implies a \times c \leq b \times c$ . Hence the system  $(\mathbb{R}, \times, \leq)$  is an isotone commutative semigroup.

Let us comment on some properties of  $\mathbb{R}_E$  and compare them with the properties of a linearly ordered field  $\mathbb{R}_D$ . Both  $\mathbb{R}_E$  and  $\mathbb{R}_D$  are ordered additive abelian groups and multiplicative commutative semigroups. The system  $\mathbb{R}_E$  does not possess an identity (one, unit-element) with respect to e-multiplication (9). Indeed, 1 is not an identity of  $\mathbb{R}_E$ , for  $1 \times a = 0 \neq a$  whenever  $a < 0$ . The distributive law does not generally hold in  $\mathbb{R}_E$ , indeed, e. g.  $(-1+2) \times 3 = 1 \times 3 = 3$ ,  $-1 \times 3 + 2 \times 3 = 0 + 6 = 6$ . Hence,  $\mathbb{R}_E$  is not a ring (without identity). Both arithmetic operations (addition and multiplication) of the system  $\mathbb{R}_E$  are order (inclusion) isotone, that is:  $a \leq b \implies a+c \leq b+c$ ,  $a \leq b \implies a \times c \leq b \times c$ . Note that the ordered field  $\mathbb{R}_D$  does not possess the latter property; namely multiplication is not inclusion isotone, as we have  $a \leq b \implies ac \leq bc$  only if  $c \geq 0$ . Inclusion isotonicity of multiplication is practically important in interval arithmetic; in fact this is the reason to consider the e-multiplication (9). Besides inclusion isotonicity another nice property of e-multiplication (9) is  $-(a \times b) = (-a) \times (-b)$ ; note that the latter does not hold for the linear multiplication. However, again we pay a price for these nice properties of  $\mathbb{R}_E = (\mathbb{R}, +, \times, \leq)$ : multiplication is distributive generally just over nonnegative summands:  $(a+b) \times c = a \times c + b \times c$ ,

$\sigma(a) = \sigma(b) = +$ . This implies that for any  $a, b, c \in \mathbb{R}_E$  we have, again similar to (5):

$$(a + b) \times c_{\sigma(a+b)} = a \times c_{\sigma(a)} + b \times c_{\sigma(b)}. \quad (11)$$

**Remark.** Instead of (9) one can define an inner multiplication by means of  $a \times' b = |a|b$ . This multiplication is also isotone and associative, but not commutative, i. e.  $(\mathbb{R}_E, \times', \leq)$  is an isotone semigroup.

From the system  $\mathbb{R}_E = (\mathbb{R}, +, \times, \leq)$  one can pass to a familiar ring. Indeed, one can define in  $\mathbb{R}_E$  the following multiplication:

$$a \cdot b = a_{\sigma(b)} \times b_{\sigma(a)} = \begin{cases} a \times b, & \text{if } a \geq 0, b \geq 0, \\ (-a) \times b, & \text{if } a \geq 0, b < 0, \\ a \times (-b), & \text{if } a < 0, b \geq 0, \\ (-a) \times (-b), & \text{if } a < 0, b < 0. \end{cases}$$

It can be checked that “ $\cdot$ ” as defined above is the familiar linear multiplication, resp. the system  $(\mathbb{R}, +, \cdot, \leq)$  is the familiar ordered real field.

As each one of the multiplications “ $\cdot$ ”, “ $\times$ ” can be defined via the other one, we shall further assume that the system  $\mathbb{R}_E$  possesses both multiplications:  $\mathbb{R}_E = (\mathbb{R}, +, \cdot, \times, \leq)$ ; this allows us to use mixed notations in expressions, e. g.  $a \cdot b + a \times b$ .

**Quasivector and e-multiplication are compatible.** In the system  $\mathbb{R}_E = (\mathbb{R}, +, \times, \leq)$  one can introduce quasivector multiplication by scalar (3), that is  $\gamma * a = |\gamma|a$ . Then one can ask if e-multiplication “ $\times$ ” defined by (9) and quasivector multiplication by scalars are compatible; indeed we have:

**Proposition 1.** *The operations e-multiplication “ $\times$ ” defined by (9) and quasivector multiplication by scalars  $\gamma * a = |\gamma|a$  are compatible, i. e.  $(\gamma * a) \times b = \gamma * (a \times b)$  holds for all  $\gamma \in \mathbb{R}_D, a, b \in \mathbb{R}$ .*

Thus we obtain a system  $(\mathbb{R}, +, \times, \mathbb{R}_D, *, \subseteq)$  which extends both the system  $\mathbb{R}_E$  and the system  $\mathbb{R}_{ED} = \mathbb{R}_{ED}^1 = (\mathbb{R}, +, \mathbb{R}_D, *, \subseteq)$ .

**Multiplicative identity.** There is no multiplicative identity in  $\mathbb{R}_E = (\mathbb{R}, +, \times, \leq)$ , so there is no point to speak of multiplicative inverse (reciprocal) and division in  $\mathbb{R}_E$ . However, there are algebraic constructions allowing to endow  $\mathbb{R}_E$  with identity. Namely, we can embed  $\mathbb{R}_E$  in the direct sum  $\mathbb{R}_D \oplus \mathbb{R}_E$  of the real field  $\mathbb{R}_D$  and  $\mathbb{R}_E$ ; then there exist various possibilities to define a product in  $\mathbb{R}_D \oplus \mathbb{R}_E$ . We shall define a product useful for interval arithmetic in the sequel.



## 2.2 Interval Arithmetic in Midpoint-radius Form

For our purposes we need to briefly outline the arithmetic operations for intervals represented in midpoint-radius form. In interval arithmetic, like in the real one, one uses two basic groups of operations:

- i) for  $n$ -tuples of intervals: addition, subtraction and multiplication by scalars; and
- ii) for one-dimensional intervals: addition, subtraction, multiplication and division.

The first group of operations leads to vector-space-like systems (modules), whereas the second one leads to specific ring-like systems (ringoids). Below we briefly present these two groups of operations/systems. Our presentation is based on a number of known results about: i) midpoint-radius form and centred intervals, see e. g. [31], [32], [33], [36], [37], and ii) generalized intervals [4], [8], [38].

One may also mention operations for matrices of intervals: addition, subtraction, multiplication by scalars and (inner) multiplication; however, these operations are heavily based on the above mentioned ones.

### 2.2.1 Vector-space-like interval arithmetic systems

For  $a', a'', x \in \mathbb{R}^n$ ,  $a'' \geq 0$ , the point set  $A = \{x \in \mathbb{R}^n \mid a' - a'' \leq x \leq a' + a''\} = \{x \in \mathbb{R}^n \mid |a' - x| \leq a''\}$  is called an ( $n$ -dimensional) *interval (box)* in  $\mathbb{R}^n$ . The interval is determined by the two  $n$ -tuples  $a', a'' \in \mathbb{R}^n$ ; hence we can write  $A = (a'; a'')$ , thereby  $a'$  is the *midpoint (center)* of  $A$  and  $a'' \geq 0$  is the *radius (error)* of  $A$ . The set of all  $n$ -dimensional intervals in  $\mathbb{R}^n$  is denoted  $I(\mathbb{R}^n)$ ,  $n \geq 1$ , symbolically  $I(\mathbb{R}) = \{(a'; a'') \mid a' \in \mathbb{R}^n, a'' \in \mathbb{R}^n, a'' \geq 0\}$ . If  $A = (a'; a'') \in I(\mathbb{R}^n)$  and  $a' = (a'_1, \dots, a'_n) \in \mathbb{R}^n$ ,  $a'' = (a''_1, \dots, a''_n) \in \mathbb{R}^n$ , then we write  $A = (a'; a'') = (a'_1, \dots, a'_n; a''_1, \dots, a''_n) \in I(\mathbb{R}^n)$ . Since  $I(\mathbb{R}^n) = I(\mathbb{R}) \times \dots \times I(\mathbb{R}) = (I(\mathbb{R}))^n$ , we may also write  $A = (a'; a'') = ((a'_1; a''_1), (a'_2; a''_2), \dots, (a'_n; a''_n))$ , where  $(a'_i; a''_i) \in I(\mathbb{R})$ ,  $i = 1, \dots, n$ .

*Addition* of two intervals  $A, B \in I(\mathbb{R}^n)$  is:

$$\begin{aligned} A + B &= (a'; a'') + (b'; b'') = (a' + b'; a'' + b'') \\ &= (a'_1 + b'_1, \dots, a'_n + b'_n; a''_1 + b''_1, \dots, a''_n + b''_n). \end{aligned} \quad (12)$$

*Multiplication of an  $n$ -dimensional interval by real scalars*  $* : \mathbb{R} \times I(\mathbb{R}^n) \longrightarrow I(\mathbb{R}^n)$  is defined by

$$\begin{aligned} \gamma * A &= \gamma * (a'; a'') = (\gamma a'; |\gamma| a'') \\ &= (\gamma a'_1, \dots, \gamma a'_n; |\gamma| a''_1, \dots, |\gamma| a''_n). \end{aligned} \quad (13)$$

The operator *negation*  $\neg : I(\mathbb{R}^n) \longrightarrow I(\mathbb{R}^n)$  is

$$\neg A = (-1) * A = (-a'; a'') = (-a'_1, \dots, -a'_n; a''_1, \dots, a''_n).$$

*Subtraction* is a composition of addition and negation, for  $A, B \in I(\mathbb{R}^n)$ :

$$\begin{aligned} A \neg B &= A + (\neg B) = (a'; a'') + (-b'; b'') = (a' - b'; a'' + b'') \\ &= (a'_1 - b'_1, \dots, a'_n - b'_n; a''_1 + b''_1, \dots, a''_n + b''_n). \end{aligned} \quad (14)$$

**Centred intervals.** Using interval addition (12) any interval  $(a'; a'')$  can be written as a sum  $(a'; a'') = (a'; 0) + (0; a'')$  of a *degenerated (point) interval*  $(a'; 0)$  and a *centred (origin symmetric) interval* of the form  $(0; a'')$ . Clearly, degenerated (point) intervals can be identified with real vectors, hence systems of point intervals are isomorphic to familiar systems of real vectors (or numbers in the one-dimensional case), such as vector spaces and rings. We stop our attention to the set of all centred intervals  $I_S(\mathbb{R}^n) = \{(0; a'') \mid a'' \in (\mathbb{R}^+)^n\}$ . Substituting  $a' = 0$  in (12), (13), we obtain the following expressions for these operation which regard to centred  $n$ -dimensional intervals  $A = (0; a''), B = (0; b'') \in I_S(\mathbb{R}^n)$ :

$$A + B = (0; a'' + b'') = (0; a''_1 + b''_1, \dots, a''_n + b''_n), \quad (15)$$

$$\gamma * A = (0; |\gamma|a'') = (0; |\gamma|a''_1, \dots, |\gamma|a''_n). \quad (16)$$

Clearly, centred  $n$ -dimensional intervals  $A = (0; a'')$  can be identified with  $n$ -tuples of nonnegative reals  $a'' \in (\mathbb{R}^+)^n$ . Hence, the study of the set of all centred  $n$ -dimensional intervals  $A = (0; a''), a'' \geq 0$ , with operations (15), (16) is reduced to the study of the system  $((\mathbb{R}^+)^n, +, \mathbb{R}_D, *)$ , with operations (1), (3).

**Generalized (Kaucher) centred intervals.** It is natural to embed the system  $((\mathbb{R}^+)^k, +, \mathbb{R}_D, *)$  in the canonical quasivector space  $(\mathbb{R}^k, +, \mathbb{R}_D, *)$  considered in Section 2.1.1. Consider now the set of generalized centred intervals  $\mathbb{I}_S(\mathbb{R}^k) = \{(0; a'') \mid a'' \in \mathbb{R}^k\}$  with operations (15), (16). Clearly, we have:

**Proposition 2.** *The system of generalized centred intervals  $(\mathbb{I}_S(\mathbb{R}^k), +, \mathbb{R}_D, *)$  is isomorphic to  $(\mathbb{R}^k, +, \mathbb{R}_D, *)$ .*

**Generalized (Kaucher) intervals.** Consider the direct sum of the vector space  $(\mathbb{R}^k, +, \mathbb{R}_D, \cdot)$  and the canonical quasivector space  $(\mathbb{R}^k, +, \mathbb{R}_D, *)$ . The elements of  $(\mathbb{R}^k, +, \mathbb{R}_D, \cdot) \oplus (\mathbb{R}^k, +, \mathbb{R}_D, *)$  are  $2k$ -tuples, of the form

$(\lambda_1, \dots, \lambda_k; \mu_1, \dots, \mu_k)$ ; they can be identified with generalized intervals, thereby  $(\lambda_1, \dots, \mu_k)$  is interpreted as a midpoint and  $(\lambda_1, \dots, \mu_k)$  — as a radius (error). Denote the set of all generalized (Kaucher) intervals by  $\mathbb{I}(\mathbb{R}^k) = \{(a'; a'' \mid a', a'' \in \mathbb{R}^k)\}$ .

Addition and multiplication by scalars in the direct sum  $(\mathbb{R}^k, +, \mathbb{R}_D, \cdot) \oplus (\mathbb{R}^k, +, \mathbb{R}_D, *)$  are defined as usually, that is, formulae (12)–(13) retain the same form, just the restriction for nonnegativity of the error components is dropped.

**Proposition 3.** *The generalized interval arithmetic system  $(\mathbb{I}(\mathbb{R}^k), +, \mathbb{R}_D, *)$  is isomorphic to the direct sum of the systems  $(\mathbb{R}^k, +, \mathbb{R}_D, \cdot)$  and  $(\mathbb{R}^k, +, \mathbb{R}_D, *)$ , symbolically  $(\mathbb{I}(\mathbb{R}^k), +, \mathbb{R}_D, *) \cong (\mathbb{R}^k, +, \mathbb{R}_D, \cdot) \oplus (\mathbb{R}^k, +, \mathbb{R}_D, *)$ .*

### 2.2.2 Ring-like interval arithmetic systems

Addition of two one-dimensional intervals  $A, B \in I(\mathbb{R})$  is a special case of (12):

$$A + B = (a'; a'') + (b'; b'') = (a' + b'; a'' + b''). \quad (17)$$

(Inner) multiplication (sometimes called *set-theoretic multiplication*) is defined for two one-dimensional intervals  $A, B \in I(\mathbb{R})$  by the expression:  $A \times B = \{c \mid c = ab, a \in A, b \in B\}$ . To present it in midpoint-radius form [10], [15], [16], [33] denote  $A = (a'; a''), B = (b'; b'') \in I(\mathbb{R})$  then the product  $P = A \times B$  is defined by

$$P = \begin{cases} (a'b' + \text{sign}(a'b')a''b''; |a'|b'' + |b'|a''), & \text{if } \kappa(A) \leq 1, \kappa(B) \leq 1; \\ \sigma(b')|B| * (a'; a''), & \text{if } C(A, B); \\ \sigma(a')|A| * (b'; b''), & \text{if } C(B, A). \end{cases} \quad (18)$$

where  $|A| = |a'| + a''$  and the functional  $\kappa$  is defined for  $A = (a'; a'') \in I(\mathbb{R})$ ,  $a' \neq 0$  by [10]:

$$\kappa(A) = a''/|a'|, \quad a' \neq 0. \quad (19)$$

Condition  $C(A, B)$ , resp.  $C(B, A)$ , comprises the cases when the interval  $A$ , resp.  $B$ , contains zero:

$$C(A, B) \iff \kappa(A) > 1 \text{ and } \kappa(A) \geq \kappa(B) \text{ or } a' = 0.$$

Note that in these cases multiplication of two intervals reduces to (the simpler) multiplication by scalars. The symbol "\*" in (18) means multiplication by scalars as in (13) in the one-dimensional case, that is  $\gamma * A = \gamma * (a'; a'') = (\gamma a'; |\gamma| a'')$ .

Let us also mention one more definition for multiplication of one-dimensional intervals. The *centered outer multiplication (co-multiplication)* in midpoint-radius form is defined for  $A, B \in I(\mathbb{R})$  as follows:

$$A \circ B = (a'; a'') \circ (b'; b'') = (a'b'; |a'b'' + |b'a'' + a''b''), \quad (20)$$

cf. e. g. [5], [10], [15], [16], [31], [32], [33], [36].

We have  $A \times B \subseteq A \circ B$ ,  $A, B \in I(\mathbb{R})$ . For a comparative study of the two multiplications (18), (20), see [10], [32]. The multiplication (20) has some shortcomings in comparison to (18); as shown in [16] (20) is not inverse inclusion isotone whereas (18) is.

Further: We shall also show that the generalized centred multiplication is not associative, whereas the generalized multiplication based on (18) is, cf. [7].

For real numbers we use the familiar preceding order “ $\leq$ ”, arriving thus to the ordered real field  $(\mathbb{R}, +, \cdot, \leq)$ . In the case of intervals (at least) three order relations are practically important, we mention the following ones:

The inclusion relation “ $\subseteq$ ” defined by

$$A \subseteq B \iff |b' - a'| \leq b'' - a'' \quad (21)$$

plays substantial roles in the algebraic constructions.

The preceding relation “ $\preceq$ ” is defined by

$$A \preceq B \iff |a'' - b''| \leq b' - a'. \quad (22)$$

We shall also make use of the component-wise order relation

$$A \leq B \iff a' \leq b' \text{ and } a'' \leq b''. \quad (23)$$

**Centred intervals.** Substituting  $a' = 0$  in (17)–(21), we obtain formulae for one-dimensional centred intervals  $A = (0; a'')$ ,  $B = (0; b'') \in I(\mathbb{R})$ :

$$A + B = (0; a'' + b''), \quad (24)$$

$$A \times B = A \circ B = (0; a''b''), \quad (25)$$

$$A \subseteq B \iff a'' \leq b''. \quad (26)$$

**Generalized (Kaucher) centred intervals.** As before we start from the abelian group  $(\mathbb{I}_G(\mathbb{R}), +)$  of generalized centred intervals with addition,

this time restricting ourselves to one-dimensional intervals. Recall that the system  $(\mathbb{I}_S(\mathbb{R}), +)$  is isomorphic to  $(\mathbb{R}, +)$ .

We now want to involve multiplication, to this end we need to extend (25) for generalized (Kaucher) centred intervals. Consider the following operation for  $(0; a''), (0; b'') \in \mathbb{I}_S(\mathbb{R})$ :

$$(0; a'') \times (0; b'') = (0; a'' \times b''), \quad (27)$$

where  $a'' \times b''$  is given by (9). For example:  $(0; 2) \times (0; 3) = (0; 6)$ ,  $(0; -2) \times (0; 3) = (0; 0)$ ,  $(0; -2) \times (0; -3) = (0; -6)$ . In end-point notation:  $[-2, 2] \times [-3, 3] = [-6, 6]$ ,  $[2, -2] \times [-3, 3] = [0, 0]$ ,  $[2, -2] \times [3, -3] = [6, -6]$ .

Consider the system  $\mathbb{R}_E = (\mathbb{R}, +, \times, \leq)$ , where “ $\times$ ” is defined by (9) and addition and order are as usual. The next Proposition gives a relation between  $\mathbb{R}_E$  and the system  $(\mathbb{I}_S(\mathbb{R}), +, \times, \subseteq)$  of generalized centred intervals, cf. [4], [6], [8], [23].

**Proposition 4.** *The system  $\mathbb{R}_E = (\mathbb{R}, +, \times, \leq)$  is isomorphic to the system  $(\mathbb{I}_S(\mathbb{R}), +, \times, \subseteq)$  of generalized centred intervals on the real line.*

Proof. Generalized centred intervals  $(0; r) \sim [-r, r]$ ,  $r \in \mathbb{R}$ , are elements of the group extension of the monoid of (proper) centred intervals. They are represented by real numbers  $r \in \mathbb{R}$ , and for  $r \geq 0$  are interpreted as radii of the centred intervals. The product of two generalized centred intervals of the form  $(0; a)$ ,  $(0; b)$  according to (27) is  $(0; a \times b)$ , where  $a \times b$  is given by (9); thereby relation “ $\leq$ ” corresponds to inclusion.  $\square$

Inclusion of generalized centred intervals is defined by means of (26) after dropping the restriction  $a'' \geq 0$ ,  $b'' \geq 0$ . For example:  $(0; -3) \subseteq (0; -1) \subseteq (0; 0) \subseteq (0; 1)$ ; in end-point form:  $[3, -3] \subseteq [1, -1] \subseteq [0, 0] \subseteq [-1, 1]$ . This order corresponds to the natural order in the set of reals.

Multiplication (27) of generalized centred intervals based on (9) is inclusion isotone, that is, for any  $a, b, c \in \mathbb{R}$

$$(0; a) \subseteq (0; b) \implies (0; c) \times (0; a) \subseteq (0; c) \times (0; b).$$

In end-point notation we can write:

$$[-a, a] \subseteq [-b, b] \implies [-c, c] \times [-a, a] \subseteq [-c, c] \times [-b, b].$$

**Generalized (Kaucher) intervals.**

Denote  $|a| = |a'| + |a''|$ . The functional  $\kappa : \mathbb{I}(\mathbb{R}) \rightarrow \mathbb{R}$  is defined for  $a = (a'; a'') \in \mathbb{I}(\mathbb{R})$  with  $a' \neq 0$  by:

$$\kappa(a) = \begin{cases} |a''|/|a'|, & \text{if } a' \neq 0, \\ \infty, & \text{if } a' = 0. \end{cases} \quad (28)$$

Using  $\kappa$  we can write the condition  $|a''| \leq |a'|$  (meaning that the interval  $a = (a'; a'')$  does not contain zero as interior point) in the form  $\kappa(a) \leq 1$ .

There is no multiplicative identity in  $\mathbb{R}_E = (\mathbb{R}, +, \times, \leq)$ , so there is no point to speak of multiplicative inverse (reciprocal) and division in  $\mathbb{R}_E$ . However, there are algebraic constructions allowing to endow  $\mathbb{R}_E$  with identity. Namely, we shall embed  $\mathbb{R}_E$  in the direct sum  $\mathbb{R}_D \oplus \mathbb{R}_E$  of the real field  $\mathbb{R}_D$  and  $\mathbb{R}_E$  thereby introducing the product  $p = a \times b$  for  $a = (a'; a'')$ ,  $b = (b'; b'') \in \mathbb{R}_D \oplus \mathbb{R}_E$  by means of:

$$p = \begin{cases} (a'b' + \text{sign}(a'b')a''b''; |a'|b'' + |b'|a''), & \text{if } \kappa(a) \leq 1, \kappa(b) \leq 1; \\ \sigma(b')|b| * a, & \text{if } \text{Cond}(a, b) \text{ holds true}; \\ \sigma(a')|a| * b, & \text{if } \text{Cond}(b, a) \text{ holds true}; \\ (0; 0), & \text{if } \sigma(a'') \neq \sigma(b''), \kappa(a) > 1, \kappa(b) > 1, \end{cases} \quad (29)$$

wherein condition  $\text{Cond}(a, b)$  is

$$\text{Cond}(a, b) \iff \sigma(a'') = \sigma(b''), \kappa(a) > 1 \text{ and } \kappa(a) \geq \kappa(b).$$

Multiplication (29) has been introduced in by E. Kaucher [8]. The endpoint form used by Kaucher is equivalent to our midpoint-radius form (29), see Appendix 1. Note that in the case when some of the conditions  $\kappa(a) \leq 1, \kappa(b) \leq 1$  is violated (that is, some of the intervals contain zero), the product  $p = a \times b$  reduces to multiplication by scalars, or is equal to zero.

**Proposition 5.** *The system  $\mathbb{R}_D \oplus \mathbb{R}_E$  has a multiplicative identity  $1 = (1; 0)$  with respect to the e-multiplication “ $\times$ ” defined by (29). Moreover, every element  $a = (a'; a'') \in \mathbb{R}_D \oplus \mathbb{R}_E$ , such that  $a' \neq 0$ , has a multiplicative inverse (reciprocal)  $a^{-1} = (1/a'; (-a'')/|a'|^2)$ .*

Further, one can introduce quasivector multiplication by scalar in  $\mathbb{R}_D \oplus \mathbb{R}_E$  by  $\gamma * (a'; a'') = (\gamma a'; |\gamma| a'')$ .

The system  $\mathbb{R}_D \oplus \mathbb{R}_E$  reflects many properties of approximate numbers (intervals on the real line).

Formula (29) is not the unique possibility to introduce identity. Another possible way to define the product of  $(a'; a''), (b'; b'') \in \mathbb{R}_D \oplus \mathbb{R}_E$  is:

$$(a'; a'') \circ (b'; b'') = (a'b'; |a'|b'' + |b'|a'' + a''b''). \quad (30)$$

It is easy to see that (30) also introduces an identity in  $\mathbb{R}_D \oplus \mathbb{R}_E$ . However, (29) is associative and inverse inclusion monotone, whereas (30) is not. This properties will be derived in Section 4.

One can consider  $m \times n$ -matrices of elements of  $\mathbb{R}_D \oplus \mathbb{R}_E$  by introducing corresponding matrix operations as done for matrices of real numbers.

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E. Kaucher gives formulae for multiplication (in end-point form) that extend (18) for generalized intervals (no restriction  $a'' \geq 0, b'' \geq 0$ ) [7]. His formulae for the product  $p = a \times b = (a'; a'') \times (b'; b'')$  of two generalized (not necessarily centred) intervals  $a = (a'; a''), b = (b'; b'') \in \mathbb{I}(\mathbb{R})$  can be presented in midpoint-radius form as follows:

$$p = \begin{cases} (a'b' + \text{sign}(a'b')a''b''; |a'|b'' + |b'|a''), & \text{if } \kappa(a) \leq 1, \kappa(b) \leq 1; \\ \sigma(b')|b| * (a'; a''), & \text{if } \text{Cond}(a, b); \\ \sigma(a')|a| * (b'; b''), & \text{if } \text{Cond}(b, a); \\ (0; 0), & \text{if } \sigma(a'') \neq \sigma(b''), \kappa(a) > 1, \kappa(b) > 1, \end{cases} \quad (31)$$

where  $|a| = |a'| + |a''|$ ,  $\kappa$  is defined for  $a = (a'; a'') \in \mathbb{I}(\mathbb{R})$  with  $a' \neq 0$  by:

$$\kappa(a) = |a''|/|a'|, \quad a' \neq 0. \quad (32)$$

and condition  $\text{Cond}(a, b)$  in (31) is

$$\text{Cond}(a, b) \iff \sigma(a'') = \sigma(b''), \kappa(a) > 1 \text{ and } \kappa(a) \geq \kappa(b) \text{ or } a' = 0.$$

The above condition corresponds to cases when  $a$  (and  $b$ ) contain (is contained in) zero. In these cases multiplication reduces to multiplication by scalars.

With the help of the *hyperbolic product* [7]:

$$a \cdot b = (a'b' + a''b'', a'b'' + a''b')$$

the first line in formula (31) can be written:

$$p = a \times b = a_{\tau(b)} \cdot b_{\tau(a)}, \quad \text{if } \kappa(a) \leq 1, \kappa(b) \leq 1,$$

where  $\tau(a) = \sigma(a'')$ .

An extension of (20):  $(a'; a'') \circ (b'; b'') = (a'b'; |a'|b'' + |b'|a'' + a''b'')$  for  $(a'; a''), (b'; b'') \in \mathbb{I}(\mathbb{R})$  is given by:

$$(a'; a'') \circ (b'; b'') = \begin{cases} (0; 0), & \sigma(a'') \neq \sigma(b''), \kappa(a) > 1, \kappa(b) > 1; \\ (a'b'; |a'|b'' + |b'|a'' + a'' \times b''), & \text{otherwise.} \end{cases} \quad (33)$$

Inclusion of generalized intervals is defined by means of (21) after dropping the restriction  $a'' \geq 0, b'' \geq 0$ , namely

$$(a'; a'') \subseteq (b'; b'') \iff |b' - a'| \leq b'' - a''. \quad (34)$$

Multiplication (31) inclusion isotone. We have  $(a'; a'') \times (b'; b'') \subseteq (a'; a'') \circ (b'; b'')$ .

.

**Proposition 6.** *The system  $\mathbb{R}_{ED} = (\mathbb{R}, +, \times, \mathbb{R}_D, *, \subseteq)$  is isomorphic to the system  $(\mathbb{I}_S(\mathbb{R}), +, \times, \mathbb{R}_D, *, \subseteq)$ .*

**Proposition 7.** *The system  $\mathbb{R}_D \oplus \mathbb{R}_{ED}$  is isomorphic to the system  $(\mathbb{I}(\mathbb{R}), +, \times, \mathbb{R}_D, *, \subseteq)$  with multiplication “ $\times$ ” defined by (33) and multiplication by scalars  $\gamma * a = |\gamma|a$ .*

We notice that the operation (33) coincides with (30). This shows the advantage of multiplication (33) being more simple than (31) from algebraic point of view.

Clearly, everything said for the interval-arithmetic system  $(\mathbb{I}(\mathbb{R}), +, \times, \subseteq)$  can be re-formulated for the system  $(\mathbb{I}(\mathbb{R}^{m \times n}), +, \times, \subseteq)$  of interval matrices under a suitable definition of matrix operations— analogously to the case with real numbers.

Our next aim is to put generalized interval arithmetic systems like  $(\mathbb{I}(\mathbb{R}^k), +, \mathbb{R}_D, *)$ ,  $(\mathbb{I}(\mathbb{R}), +, \times, \subseteq)$ ,  $(\mathbb{I}(\mathbb{R}), +, \times, \mathbb{R}_D, *, \subseteq)$ ,  $(\mathbb{I}(\mathbb{R}^{m \times n}), +, \times, \subseteq)$  on an axiomatic foundation.

## 3 Quasivector Spaces

### 3.1 Quasivector Spaces: Definition

**Definition 1.** *A quasivector space (over the l. o. field  $\mathbb{R}_D = (\mathbb{R}, +, \cdot, \leq)$ ), denoted  $(\mathcal{Q}, +, \mathbb{R}_D, *)$ , is an abelian group  $(\mathcal{Q}, +)$  with a mapping (multiplication by scalars) “ $*$ ”:  $\mathbb{R} \times \mathcal{Q} \rightarrow \mathcal{Q}$ , such that for  $a, b, c \in \mathcal{Q}$ ,  $\alpha, \beta, \gamma \in \mathbb{R}$ :*

$$\gamma * (a + b) = \gamma * a + \gamma * b, \quad (35)$$

$$\alpha * (\beta * c) = (\alpha\beta) * c, \quad (36)$$

$$1 * a = a, \quad (37)$$

$$(\alpha + \beta) * c = \alpha * c + \beta * c, \quad \text{if } \alpha\beta \geq 0. \quad (38)$$



**Conjugate elements.** Denote  $\neg a = (-1) * a$ . By definition quasivector spaces are groups, hence the existence of opposite elements  $\text{opp}(a)$  is assumed. From  $a + \text{opp}(a) = 0$  we obtain  $\neg \text{opp}(a) \neg a = 0$ , that is  $\neg \text{opp}(a) = \text{opp}(\neg a)$ . The element  $\neg \text{opp}(a) = \text{opp}(\neg a)$  is further denoted by  $a_-$  and the corresponding operator is called *dualization* or *conjugation*. We say that  $a_-$  is the conjugate (or dual) of  $a$ . Relations  $\neg \text{opp}(a) = \text{opp}(\neg a) = a_-$  imply  $\text{opp}(a) = \neg(a_-) = (\neg a)_-$ , which is denoted  $\text{opp}(a) = \neg a_-$ .

**Remark.** Note that in Definition 1 we do not assume  $\neg a = a$  and hence  $\text{opp}(a) = a_-$  may not hold true in a general quasivector space (as was the case with the space  $(\mathbb{R}^k, +, \mathbb{R}_D, *)$ , resp.  $(\mathbb{I}_S(\mathbb{R}^k), +, \mathbb{R}_D, *)$  wherein the relation  $\neg a = a$  holds for all elements  $a$ ).

**The quasidistributive law.** Rules for calculation in quasivector spaces are derived in [12], [13], [18]. Here we shall pay attention to the quasidistributive law. The condition  $\alpha\beta \geq 0$  in (38) creates the false impression that there may be some freedom in the form of the distributivity relation for  $\alpha\beta < 0$ . The following result shows that this is not the case: it turns out that (38) determines a specific relation for all  $\alpha, \beta \in \mathbb{R}$ .

**Proposition 8.** *Let  $(\mathcal{Q}, +, \mathbb{R}, *)$  be a quasivector space over  $\mathbb{R}$ . For  $\alpha, \beta \in \mathbb{R}$  and  $c \in \mathcal{Q}$  we have:*

$$(\alpha + \beta) * c_{\sigma(\alpha+\beta)} = \alpha * c_{\sigma(\alpha)} + \beta * c_{\sigma(\beta)}. \quad (39)$$

**Definition 2.**  *$\mathcal{Q}$  is a quasivector space. An element  $a \in \mathcal{Q}$  with  $a \neg a = 0$  is called linear. An element  $a \in \mathcal{Q}$  with  $\neg a = a$  is called origin symmetric or centred.*

Note that the relation  $a \neg a = 0$  is equivalent to  $a = a_-$ , and the relation  $\neg a = a$  is equivalent to  $a + a_- = 0$ . The latter means that, if  $a$  is centred, then  $\text{opp}(a) = a_-$ . For any  $a$  the element  $a + a_-$  is linear.

Subspaces, sum and direct sum of quasivector spaces are defined as in vector spaces. It can be checked that in a quasivector space  $\mathcal{Q}$  the subset of all linear elements  $\mathcal{Q}' = \{a \in \mathcal{Q} \mid a \neg a = 0\}$  forms a subspace of  $\mathcal{Q}$ ; so does the subset of all centred elements  $\mathcal{Q}'' = \{a \in \mathcal{Q} \mid a = \neg a\}$ .

**Proposition 9.** *Assume that  $\mathcal{Q}$  is a quasivector space. The subspace  $\mathcal{Q}' = \{a \in \mathcal{Q} \mid a \neg a = 0\}$  is a vector space.*

Indeed, we only have to check that relation (38) becomes true for all values of the scalars. However, this is obvious from (39) as linear elements satisfy  $c = c_-$ .

**Definition 3.** Assume that  $\mathcal{Q}$  is a quasivector space. The space  $\mathcal{Q}' = \{a \in \mathcal{Q} \mid a \neg a = 0\}$  is called the linear subspace of  $\mathcal{Q}$  and the space  $\mathcal{Q}'' = \{a \in \mathcal{Q} \mid a = \neg a\}$  is called the centred (quasivector) subspace of  $\mathcal{Q}$ .

**Theorem 1. (Decomposition theorem)** Every quasivector space  $\mathcal{Q}$  is a direct sum of its linear and centred subspaces, i. e.  $\mathcal{Q} = \mathcal{Q}' \oplus \mathcal{Q}''$ . More specifically, for every  $x \in \mathcal{Q}$  we have  $x = x' + x''$  with unique  $x' = (1/2) * (x + x_-) \in \mathcal{Q}'$ , and  $x'' = (1/2) * (x \neg x) \in \mathcal{Q}''$ .

**Proof.** Assume  $x \in \mathcal{Q}$ . Using that  $x_- \neg x = 0$  we have

$$\begin{aligned} x' + x'' &= (1/2) * (x + x_-) + (1/2) * (x \neg x) \\ &= (1/2) * (x + x + x_- \neg x) = x. \end{aligned}$$

On the other side we have  $x' = (1/2) * (x + x_-) \in \mathcal{Q}'$  and  $x'' = (1/2) * (x \neg x) \in \mathcal{Q}''$ . Hence,  $\mathcal{Q} = \mathcal{Q}' + \mathcal{Q}''$ . Furthermore,  $\mathcal{Q}' \cap \mathcal{Q}'' = 0$ . Indeed, assume  $x \in \mathcal{Q}'$  and  $x \in \mathcal{Q}''$ . Then we have simultaneously  $x \neg x = 0$  and  $x = \neg x$ , implying  $x = 0$ . Hence  $\mathcal{Q} = \mathcal{Q}' \oplus \mathcal{Q}''$ .  $\square$

Theorem 1 states that every element  $x \in \mathcal{Q}$  can be decomposed in a unique way as  $x' + x''$ , where  $x'$  is an element of a vector space and  $x''$  belongs to a centred quasivector space. We call  $x'$  the *linear component* of  $x$ , and  $x''$  — the *centred component* of  $x$ , and write  $x = (x'; x'')$ .

Theorem 1 shows that the midpoint-radius form is the natural form for algebraic point of view. This is the reason to use it for the presentation of interval arithmetic in Section 2. Let us mention that there exist also other purely practical motivations in favor to midpoint-radius form [32], [35].

**Theorem 2.** Let  $(\mathcal{Q}, +, \mathbb{R}, *)$  be a quasivector space over  $\mathbb{R}$ . Then  $(\mathcal{Q}, +, \mathbb{R}, \cdot)$ , with “ $\cdot$ ”:  $\mathbb{R} \times \mathcal{Q} \rightarrow \mathcal{Q}$  defined for  $\alpha \in \mathbb{R}$ ,  $c \in \mathcal{Q}$  by

$$\alpha \cdot c = \alpha * c_{\sigma(\alpha)} = \begin{cases} \alpha * c, & \text{if } \alpha \geq 0, \\ \alpha * c_-, & \text{if } \alpha < 0, \end{cases} \quad (40)$$

is a vector space over  $\mathbb{R}$ .

Relation (40) shows that the linear multiplication “ $\cdot$ ” is an inherent operation for every quasilinear space (not necessarily centred); although it is not explicitly included in the notation  $(\mathcal{Q}, +, \mathbb{R}, *)$ , it is present there in the same way as the operators opposite, negation and conjugation are present.

The question arises: is it possible to represent all quasilinear operations and expressions in a quasivector space by means of linear operations?

The answer to the above question is generally “no”, that is, we cannot generally represent in linear terms the quasivector multiplication by scalars. Indeed, from (40) we have

$$\alpha * c = \alpha \cdot c_{\sigma(\alpha)} = \begin{cases} \alpha \cdot c, & \text{if } \alpha \geq 0, \\ \alpha \cdot c_-, & \text{if } \alpha < 0. \end{cases} \quad (41)$$

Relation (41) shows that the quasivector expression  $\alpha * c$  is presented as  $\alpha \cdot c_{\sigma(\alpha)}$  using the linear multiplication and the operator conjugation. Clearly, the “quasivector” operations “ $*$ ”, “ $-$ ” and “ $c_-$ ” cannot be generally represented by means of linear operations. However, in what follows we are going to show that such a representation does exist in the special case of centred quasivector spaces.

### 3.2 Centred Quasivector Spaces and E-linear spaces

Theorem 1 states that every quasivector space is a direct sum of a vector space and a centred quasivector space. As vector spaces are well-known, we need to study centred quasivector spaces.

A centred quasivector space  $\mathcal{S}$  can be defined axiomatically as an abelian group with multiplication by scalars (from a l. o. field) satisfying (35)–(38) together with the additional assumption:  $\neg a = a$  for all  $a \in \mathcal{S}$ . Starting from this axiomatic definition one can develop the theory of centred quasivector spaces following step by step the development of the theory of vector spaces. This approach has been exploited in [12]. We now concentrate on the relations between quasivector and linear multiplication by scalars; then the properties of centred quasivector spaces become transparent.

**The centred case.** In a centred quasivector space, due to  $(-1) * c = c$ , we have:  $\alpha * c = (-\alpha) * c = |\alpha| * c$ . Hence formula (40) for the induced linear multiplication in a centred quasivector space can be written as

$$\alpha \cdot c = |\alpha| * c_{\sigma(\alpha)}. \quad (42)$$

Recall the relation  $\text{opp}(c) = c_-$ , which is true for centred elements and not true in a general quasivector space. This relation shows that, in the case of centred quasivector space, the operation conjugation can be presented by the linear operation “opposite”. Substituting  $c_- = \text{opp}(c)$  in (41) gives

$$\alpha * c = |\alpha| \cdot c. \quad (43)$$

Relation (43) shows that in a centred quasivector space the quasivector multiplication “ $*$ ” is representable by the linear multiplication and the function modulus. In particular, negation is representable by linear operations, due to  $\neg c = c$ . (Note that the above is not true in a general quasivector space, but is true in a centred quasivector space.) We thus obtain the following special case of Theorem 2:

**Corollary 1.** Let  $(\mathcal{S}, +, \mathbb{R}, *)$  be a centred quasivector space over  $\mathbb{R}$ . The induced vector space  $(\mathcal{S}, +, \mathbb{R}, \cdot)$ , with “ $\cdot$ ” defined by (42), due to  $c_- = \text{opp}(c)$ , involves implicitly quasilinear multiplication by scalars, and hence all (quasivector) operations/expressions in  $(\mathcal{S}, +, \mathbb{R}, *)$  can be presented by linear ones.

**Corollary 2.** Every vector space over a l. o. field  $\mathbb{R}$  induces via (43) a centred quasivector space.

Thus to every vector space over a l. o. field  $(\mathcal{G}, +, \mathbb{R}, \cdot)$ , we associate the centred quasivector space  $(\mathcal{G}, +, \mathbb{R}, *)$  with “ $*$ ” defined by (43). The two spaces —  $(\mathcal{Q}, +, \mathbb{R}, *)$  and  $(\mathcal{Q}, +, \mathbb{R}, \cdot)$  — differ from each other by having different operations for multiplication by scalars.

To summarize: *Every centred quasivector space over  $\mathbb{R}$  generates via (42) a linear multiplication by scalars and hence a vector space. Vice versa, every vector space over  $\mathbb{R}$  induces via (43) a quasilinear multiplication by scalars and thus a centred quasivector space.*

Let  $(\mathcal{S}, +, \mathbb{R}, *)$  be a *centred* quasivector space and  $(\mathcal{S}, +, \mathbb{R}, \cdot)$  be the associated vector space. It follows from the above, that all concepts characteristic for the vector space  $(\mathcal{S}, +, \mathbb{R}, \cdot)$ , such as linear combination, linear dependence, basis etc., can be represented in terms of the original centred quasivector space  $(\mathcal{S}, +, \mathbb{R}, *)$ . We thus feel free to use any vector space concepts in the centred quasivector space.

As every quasivector space  $\mathcal{Q}$  is a direct sum  $\mathcal{Q} = \mathbb{V} \oplus \mathcal{S}$  of a vector space  $\mathbb{V}$  and a centred quasivector space  $\mathcal{S}$ , we can speak of basis and dimension of  $\mathcal{Q}$ , whenever  $\mathbb{V}$  and  $\mathcal{S}$  have finite bases. Namely, let  $\mathbb{V} = \mathbb{V}^l$  be a  $l$ -dimensional vector space with a basis  $(v^{(1)}, \dots, v^{(l)})$  and let  $\mathcal{S} = \mathbb{S}^k$  be a  $k$ -dimensional centred quasivector space having a basis  $(s^{(1)}, \dots, s^{(k)})$ . Then we say that  $(v^{(1)}, \dots, v^{(l)}; s^{(1)}, \dots, s^{(k)})$  is a basis of the  $(l, k)$ -dimensional quasivector space  $\mathcal{Q} = \mathbb{V}^l \oplus \mathbb{S}^k$ . In interval arithmetic we usually have  $l = k$  as in the following proposition.

**Proposition 10.** *The Kaucher interval arithmetic system  $(\mathbb{I}(\mathbb{R}^k), +, \mathbb{R}_D, *) \cong (\mathbb{R}^k, +, \mathbb{R}, \cdot) \oplus (\mathbb{R}^k, +, \mathbb{R}, *)$  is a quasivector space.*

**Definition 4.** An e-linear space over the real l. o. field  $\mathbb{R}_D$  is a system  $(\mathbb{S}, +, \mathbb{R}_D, *, \subseteq)$  with the following properties: 1.  $(\mathbb{S}, \subseteq)$  is a (partially) ordered set; 2.  $(\mathbb{S}, +, \mathbb{R}_D, *)$  is a quasivector space over the l. o. field  $\mathbb{R}_D$ , and 3. addition “+” and multiplication by scalars “\*” are order isotone, that is for  $a, b, c \in \mathbb{S}$  and  $\gamma \in \mathbb{R}_D$ :  $a \subseteq b \implies a + c \subseteq b + c$  and  $\gamma * a \subseteq \gamma * b$ .

Briefly, an e-linear space is a partially ordered quasivector space.

**Proposition 11.** The Kaucher centred interval arithmetic system  $(\mathbb{I}_S(\mathbb{R}^k), +, \mathbb{R}_D, *, \subseteq)$  is an e-linear space.

**Proposition 12.** The Kaucher interval arithmetic system  $(\mathbb{I}(\mathbb{R}^k), +, \mathbb{R}_D, *, \subseteq)$  is an e-linear space.

Remark. Similar propositions hold for the set  $\mathbb{I}(\mathbb{R}^{m \times n})$  of interval matrices.

### 3.3 E-rings

We defined the systems  $\mathbb{R}_E = (\mathbb{R}, +, \times, \sqsubseteq)$ ,  $\mathbb{R}_{ED} = (\mathbb{R}, +, \times, \mathbb{R}_D, *, \sqsubseteq)$ ,  $(\mathbb{R}_D \oplus \mathbb{R}_{ED}, +, \times)$ , etc. by means of the real field  $\mathbb{R}_D$ . Our next aim is to define these systems axiomatically and independently on the definition of  $\mathbb{R}_D$ .

#### 3.3.1 Axiomatic definition of e-ring

**Definition 5.** *An e-ring is a system  $(\mathbb{S}, +, \times, \sqsubseteq)$ , such that:*

- A1.  $(\mathbb{S}, \sqsubseteq)$  is an ordered set (a poset);
- A2.  $(\mathbb{S}, +, \sqsubseteq)$  is an isotone abelian group (with null 0);
- A3. Multiplication “ $\times$ ” in  $(\mathbb{S}, +, \times, \sqsubseteq)$  is order isotone and distributive over a sum of any two proper elements (elements that  $\sqsupseteq 0$ ).

Recall that the familiar ordered field of reals  $\mathbb{R}_D = (\mathbb{R}, +, \cdot, \leq)$  is a (linearly) ordered ring. Indeed,  $\mathbb{R}_D$  satisfies assumptions A1. and A2. Concerning assumption A3.,  $(\mathbb{R}, \cdot, \leq)$  is a commutative semigroup, and multiplication “ $\cdot$ ” is distributive over the sum of any two elements. But  $(\mathbb{R}, \cdot, \leq)$  is not isotone, that is  $a \leq b \implies ac \leq bc$  does not generally hold (for all  $a, b, c$ ). In contrast  $\mathbb{S}$  assumed to be isotone. We show in the sequel that the lack of distributivity in  $\mathbb{S}$  is not a substantial drawback, by deriving a specific distributive-like relation in  $\mathbb{S}$ .

For more clarity let us write down the assumptions of an e-ring in detail. The null in the additive group  $(\mathbb{S}, +)$  is denoted as usually by 0; the opposite element to  $a \in \mathbb{S}$  is denoted symbolically by  $\text{opp}(a)$  or  $-a$ . (However, the symbol “ $-$ ” should be used carefully. In interval and convex analysis this symbol is used to denote multiplication by the scalar  $-1$ , which is generally different from opposite.)

Using these notations, according to Definition 5, a system  $(\mathbb{S}, +, \times, \sqsubseteq)$  is an *e-ring* if the following 10 axioms (classified in three groups) hold:

- A1.  $(\mathbb{S}, \sqsubseteq)$  is partially ordered, i. e. for all  $a, b, c \in \mathbb{S}$ :
  - A1.1.  $a \sqsubseteq a$ ,
  - A1.2.  $a \sqsubseteq b, b \sqsubseteq a \implies a = b$ ,
  - A1.3.  $a \sqsubseteq b, b \sqsubseteq c \implies a \sqsubseteq c$ ;

- A2.  $(\mathbb{S}, +, \sqsubseteq)$  is an isotone abelian group with null 0 and opposite  $\text{opp}(a)$ ,  
i. e. for all  $a, b, c \in \mathbb{S}$ :
- A2.1.  $(a + b) + c = a + (b + c)$ ,
- A2.2.  $a + b = b + a$ ,
- A2.3.  $a + 0 = a$ ,
- A2.4.  $a + \text{opp}(a) = 0$ ,
- A2.5.  $a \sqsubseteq b \implies a + c \sqsubseteq b + c$ ;
- A3. Multiplication “ $\times$ ” in  $(\mathbb{S}, +, \times, \sqsubseteq)$  is order isotone and distributive over  
a sum of any two proper elements, i. e. for all  $a, b, c \in \mathbb{S}$ :
- A3.1.  $a \sqsubseteq b \implies a \times c \sqsubseteq b \times c$ ;
- A3.2.  $(a + b) \times c = a \times c + b \times c$ , whenever  $0 \sqsubseteq a, b$ .

We shall consider three more additional axioms:

- A1.4. For any  $a, b$  either  $a \sqsubseteq b$  or  $b \sqsubseteq a$  holds true;
- A3.3. For any  $a, b$  relation  $a \times b = b \times a$  holds true;
- A3.4. For any  $a, b, c$  relation  $a \times (b \times c) = (a \times b) \times c$  holds true.

**Definition 6.** *An e-ring, satisfying A1.4., is called linearly ordered. An e-ring, satisfying A3.3., is called commutative. An e-ring, satisfying A3.4., is called associative. An e-ring  $\mathbb{S}$  is an e-ring with identity if there is an element  $e$  in  $\mathbb{S}$  such that  $a \times e = e \times a = a$  for all  $a \in \mathbb{S}$ .*

If there is an identity, it is clearly unique, and will be denoted, as usually, by 1. Further, as with familiar rings, if the e-ring is not the null one (that is consists only of the null element 0), then  $1 \neq 0$ .

**Definition 7.** *Let  $\mathbb{S}$  be an e-ring. If  $a \neq 0, b \neq 0$  are elements of  $\mathbb{S}$  such that  $a \times b = 0$ , then  $a$  is called left zero divisor and  $b$  is called right zero divisor.*

The following theorem gives an example of an e-ring.

**Theorem 3.** *The systems  $\mathbb{R}_E = (\mathbb{R}, +, \times, \leq) \cong (\mathbb{I}_S(\mathbb{R}), +, \times, \subseteq)$  of errors, resp. generalized symmetric intervals, with multiplication (9) are linearly ordered, commutative and associative e-rings without identity with all their elements being zero divisors.*

The verification of Theorem 3 is straightforward. Theorem 3 gives an algebraic characterization of the system of generalized centred intervals on the real line.

### 3.3.2 Properties of the e-ring

We next derive some properties of the e-ring  $(\mathbb{S}, +, \times, \sqsubseteq)$ . The following familiar relations P1.1–P1.9 take place in any isotone additive group  $(\mathbb{S}, +, \sqsubseteq)$ ; as usually “ $a \subset b$ ” means “ $a \sqsubseteq b$  and  $a \neq b$ ”.

$$\text{P1.1. } a + c = b + c \implies a = b;$$

$$\text{P1.2. } a \sqsubset b \implies a + c \sqsubset b + c;$$

$$\text{P1.3. } a + c \sqsubseteq b + c \implies a \sqsubseteq b;$$

$$\text{P1.4. } a + \text{opp}(a) = 0;$$

$$\text{P1.5. } \text{opp}(0) = 0;$$

$$\text{P1.6. } \text{opp}(\text{opp}(a)) = a;$$

$$\text{P1.7. } \text{opp}(a + b) = \text{opp}(a) + \text{opp}(b);$$

$$\text{P1.8. } a \sqsubseteq b, c \sqsubseteq d \implies a + c \sqsubseteq b + d. \text{ (Indeed, } a + c \sqsubseteq b + c \sqsubseteq b + d\text{).}$$

As a corollary we obtain:  $a \sqsubseteq 0, b \sqsubseteq 0 \implies a + b \sqsubseteq 0$ , resp.  $0 \sqsubseteq a, 0 \sqsubseteq b \implies 0 \sqsubseteq a + b$ .

P1.9.  $a \subset 0 \implies 0 \subset \text{opp}(a)$ ;  $0 \subset a \implies \text{opp}(a) \subset 0$ . (Indeed, from  $a \subset 0$ , adding  $\text{opp}(a)$  to both sides we obtain:  $a + \text{opp}(a) \subset 0 + \text{opp}(a)$ , i. e.  $0 \subset \text{opp}(a)$ .) More generally,  $a \sqsubseteq b \iff \text{opp}(b) \sqsubseteq \text{opp}(a)$ .

We next deduce consequences from axioms A3.

The following properties P2.1–P2.4 hold in any isotone (multiplicative) semigroup (groupoid?)  $(\mathbb{S}, \times, \sqsubseteq)$ :

$$\text{P2.1. } a \sqsubseteq b, c \sqsubseteq d \implies a \times c \sqsubseteq b \times d.$$

Proof. Using A3.2. twice we obtain  $a \times c \sqsubseteq b \times c \sqsubseteq b \times d$ . As a special case we obtain:

$$\text{P2.2. } a \sqsubseteq 0, b \sqsubseteq 0 \implies a \times b \sqsubseteq 0, \text{ resp. } 0 \sqsubseteq a, 0 \sqsubseteq b \implies 0 \sqsubseteq a \times b.$$

$$\text{P2.3. } a \times c = b \times c, 0 \sqsubseteq a, b, 0 \sqsubseteq c \implies a = b.$$

$$\text{P2.4. } a \times c \sqsubseteq b \times c, 0 \sqsubseteq a, b, 0 \sqsubseteq c \implies a \sqsubseteq b.$$

$$\text{P2.5. } 0 \times c = c \times 0 = 0.$$

Proof. We have  $0 + 0 = 0$ . Using assumption A3.2 we have  $(0 + 0) \times c = 0 \times c + 0 \times c$ , or  $0 \times c = 0 \times c + 0 \times c$ , implying  $0 \times c = 0$ . Relation  $c \times 0 = 0$  is proved



similarly.

$$\text{P2.6. } a \sqsubseteq 0 \sqsubseteq b \implies a \times b = 0.$$

Proof. Assume  $a \sqsubseteq 0 \sqsubseteq b$ . Using assumption A3.2. and P2.5. we obtain:  
i)  $a \sqsubseteq 0 \implies a \times b \sqsubseteq 0 \times b = 0$ , and ii)  $0 \sqsubseteq b \implies 0 = 0 \times a \sqsubseteq a \times b$ . Thus we have simultaneously the inclusions  $a \times b \sqsubseteq 0$  and  $0 \sqsubseteq a \times b$ , implying (by A1.2.)  $a \times b = 0$ .

As a particular case of P2.6. we obtain (using P1.8):

$$\text{P2.7. } a \times \text{opp}(a) = 0. \text{ In particular, } 0 \times 0 = 0 \times \text{opp}(0) = 0.$$

Remark. A possible approach to the study of an e-ring could be to start with an axiomatic definition based on the two operations (addition and multiplication) without the order relation. To this end one can use axioms A2.1., A2.2., A2.3., A2.4., A3.1., A3.3., A4.1., and, additionally properties P2.5 and P2.7 (postulated as axioms). Such a system would be the counterpart of a familiar ring (without an order). Also, one can generalize the definition of an e-ring to the noncommutative case in order to include the case of matrices of errors, by dropping assumption A3.3 related to commutativity.

We now introduce one more symbolic notation. We shall write  $a_+ = a$ , so that the symbol  $a_\lambda$  makes sense for  $\lambda \in \Lambda = \{+, -\}$ . In the set  $\Lambda$  we introduce a “product” by means of  $++ = -- = +$ ,  $+- = -+ = -$ .

Up to the end of the section we denote  $a_- = \text{opp}(a)$ .

Up to the end of the subsection we denote  $a_- = \text{opp}(a)$ .

Denote by  $\mathbb{S}_+ = \{a \in \mathbb{S} \mid a \supseteq 0\}$ ,  $\mathbb{S}_- = \mathbb{S} \setminus \mathbb{S}_+$ , the sets of *proper*, resp. *improper* elements of  $\mathbb{S}$ ; the set  $\mathbb{S}_+ \setminus \{0\}$  comprises the *strictly proper* elements of  $\mathbb{S}$ . We thus read  $a \in \mathbb{S}_+$  as “ $a$  is proper element of  $\mathbb{S}$ ”, etc. Any element of  $\mathbb{S}$  is either proper or improper. Using the sets  $\mathbb{S}_+$  and  $\mathbb{S}_-$  the condition “ $a, b \supseteq 0$ ” of assumption A3.2 can be written as “ $a, b \in \mathbb{S}_+$ ”.

We introduce a function  $\tau : \mathbb{S} \longrightarrow \Lambda = \{+, -\}$  (type, direction, orientation of an element of  $\mathbb{S}$ ) by

$$\tau(a) = \begin{cases} +, & a \in \mathbb{S}_+, \\ -, & a \in \mathbb{S}_-. \end{cases} \quad (44)$$

Clearly  $\tau(a) = \tau(b)$ , if  $a, b \in \mathbb{S}_+$  or  $a, b \in \mathbb{S}_-$ .

(The function  $\tau$  is similar to the function  $\sigma$ , but  $\tau$  is defined in an e-ring whereas  $\sigma$  is defined in a ring—the field of reals.)

**Theorem 4.** (*Quasidistributive law in an e-ring*) For any  $a, b, c$  from an e-ring we have:

$$(a + b) \times c_{\tau(a+b)} = a \times c_{\tau(a)} + b \times c_{\tau(b)}. \quad (45)$$

Proof. Consider the case  $\tau(a) = \tau(b)$ . In the subcase  $\tau(a) = \tau(b) = +$  we have  $\tau(a + b) = +$  and (45) is true for  $\tau(c) = +$  by assumption A4.2.; for  $\tau(c) = -$  all products are equal to zero (according to P3.3.) so that (45) holds. In the subcase  $\tau(a) = \tau(b) = -$  we have  $\tau(a + b) = -$ . If  $\tau(c) = -$  then the validity of (45) follows from the case  $\tau(a) = \tau(b) = \tau(c) = +$  after dualizing the relation  $(a + b) \times c = a \times c + b \times c$  and applying A4.1, P1.7 to obtain  $(a_- + b_-) \times c_- = a_- \times c_- + b_- \times c_-$ . For  $\tau(c) = +$  all products are equal to zero (according to P3.3.) so that (45) holds.

Consider now the case  $\tau(a) = -\tau(b)$ . Assume that  $0 \subseteq a, b \subseteq 0$  and  $0 \subseteq b_- \subseteq a$ . In this subcase we have  $0 \subseteq a + b$  and we can write (using A4.2.)

$$((a + b) + b_-) \times c + b \times c_- = (a + b) \times c + b_- \times c + b \times c_- = (a + b) \times c,$$

so that (45) is proved in this subcase (in the last equality we use property P3.1.). The remaining subcases are verified similarly.  $\square$

**Corollary.** We have  $(a + b) \times c = a \times c + b \times c$ , if  $a, b, c \supseteq 0$  or  $a, b, c \subseteq 0$ .

“Conjugating” both sides of (45) by  $\tau(a + b)$  we obtain

$$(a + b) \times c = a \times c_{\tau(a)\tau(a+b)} + b \times c_{\tau(b)\tau(a+b)}. \quad (46)$$

The identity (46) shows, that the quasidistributive law permits opening brackets in expressions of the form  $(a + b) \times c$  similarly to opening brackets in a ring.

Property. i)  $(a \times b)_- = a_- \times b_-$ , or  $\text{opp}(a \times b) = \text{opp}(a) \times \text{opp}(b)$

ii)  $(a \times b_-)_- = a_- \times b$

Proof. Using (45) we can write:

$$0 = a_{\tau(b+b_-)} \times (b + b_-) = a_{\tau(b)} \times b + a_{\tau(b_-)} \times b_-. \quad (44)$$

Assume  $0 \subseteq b$ , then the above yields  $0 = a \times b + a_- \times b_-$ . This means that  $a_- \times b_-$  is the opposite to  $a \times b$ , i. e.  $(a \times b)_- = a_- \times b_-$ .

Assume now  $b \subseteq 0$ , then (44) yields  $0 = a_- \times b + a \times b_-$ . This means that  $a \times b_-$  is the opposite to  $a_- \times b$ , i. e.  $(a \times b_-)_- = a_- \times b$ .

### 3.3.3 Relation between a l. o. ring and an e-ring

Formula (9) defines e-multiplication in a l. o. ring by the familiar linear multiplication. Similarly, we can define linear multiplication in an e-ring

$$a \cdot b = a_{\tau(b)} \times b_{\tau(a)}, \quad (47)$$

where  $\tau$  is defined by (44).

**Theorem 5.** *Every l. o. ring (l. o. field) generates via (9) a unique (up to isomorphism) e-ring and vice versa, every e-ring induces via (47) a unique ring.*

*Proof.* Let  $(\mathbb{S}, +, \cdot, \leq)$  be a linearly ordered ring. Define a multiplication “ $\times$ ” for elements of  $\mathbb{S}$  by (9). We have to show that  $(\mathbb{S}, +, \times, \subseteq)$  with  $\subseteq$  same as  $\leq$  (and  $\tau$  same as  $\sigma$ ) is an e-ring. We prove A3.1. If the signs of  $a, b, c$  are not equal, then, using (9), A3.1 gives  $0 = 0$ , which is true. If  $\tau(a) = \tau(b) = \tau(c) = +$ , then A3.1 becomes  $(ab)c = a(bc)$ , which is true, for  $(\mathbb{S}, +, \cdot, \leq)$  is assumed a ring. If  $\tau(a) = \tau(b) = \tau(c) = -$ , then A3.1 becomes  $-((-ab)c) = -a(-bc)$ , which is true in the ring  $(\mathbb{S}, +, \cdot, \leq)$ . To prove A4 replace (47) in the distributive law  $(a + b)c = ab + bc$ . Relations A3.2, A3.3 (and A5) are verified similarly.

Conversely, let  $(\mathbb{S}, +, \times, \subseteq)$  be an e-ring. Define  $\tau$  by (44) and a multiplication “ $\cdot$ ” by (47). We shall show that  $(\mathbb{S}, +, \cdot, \leq)$  with “ $\cdot$ ” defined by (47) and  $\leq$  same as  $\subseteq$ , is a l. o. ring. The order axioms are clearly fulfilled. We shall prove the associative law:  $(ab)c = a(bc)$ . The latter, according to (47) reads:

$$a(b_{\tau(c)} \times c_{\tau(b)}) = (a_{\tau(b)} \times b_{\tau(a)})c. \quad (48)$$

We note that  $\tau(b_{\tau(c)}) = \tau(b)\tau(c)$ , and  $\tau(c_{\tau(b)}) = \tau(b)\tau(c)$ , and therefore  $\tau(b_{\tau(c)} \times c_{\tau(b)}) = \tau(b)\tau(c)$ . Thus (48) becomes

$$a_{\tau(b)\tau(c)} \times (b_{\tau(c)} \times c_{\tau(b)})_{\tau(a)} = (a_{\tau(b)} \times b_{\tau(a)})_{\tau(c)} \times c_{\tau(a)\tau(b)},$$

which is true, according to A3.1. Let us prove now the distributive law. Using that  $(a + b)_\lambda = a_\lambda + b_\lambda$ , we see that the quasidistributive law (45):  $(a + b) \times c_{\tau(a+b)} = a \times c_{\tau(a)} + b \times c_{\tau(b)}$  is equivalent to  $(a + b)_{\tau(c)} \times c_{\tau(a+b)} = a_{\tau(c)} \times c_{\tau(a)} + b_{\tau(c)} \times c_{\tau(b)}$  and, using (47) we see that the latter is equivalent to the distributive law  $(a + b)c = ac + bc$  in the corresponding ring. The rest of the ring axioms are verified similarly.  $\square$

Let us note that in an extended ring  $(\mathbb{S}, +, \cdot, \times, \leq)$  we can use mixed notations — both from a ring and from an e-ring, e. g. formula (9) can be written

$$a \times b = \begin{cases} (ab)_\tau, & \text{if } \tau(a) = \tau(b) = \tau, \\ 0, & \text{if } \tau(a) = -\tau(b). \end{cases} \quad (49)$$

Theorem 5 shows that results from a field can be re-formulated as results in the induced e-rings by means of (47) and vice versa, results from an e-ring can be reformulated as results in the induced field by means of (9) (or (49)).

## 4 The E-algebra

In this Section we combine an e-linear space with an e-ring.

**Definition 8.** *An e-algebra is an e-ring, which is an e-linear space over the l. o. real field  $\mathbb{R}_D$ , and multiplications in both spaces are compatible.*

**Theorem 6.** *A system  $(\mathbb{S}, +, \times, \mathbb{R}_D, *, \subseteq)$  is an e-algebra, iff:*

AA1.  $(\mathbb{S}, +, \times, \subseteq)$  is an e-ring;

*bez linejnost?*

AA2.  $(\mathbb{S}, +, \mathbb{R}_D, *)$  is a centred quasivector space over the l. o. real field  $\mathbb{R}_D$ ;

*bez centred?*

AA3. For all  $\gamma \in \mathbb{R}$ ,  $a, b \in \mathbb{S}$ , we have  $(\gamma * a) \times b = \gamma * (a \times b)$ .

Proof. We have to prove that a system  $(\mathbb{S}, +, \times, \mathbb{R}_D, *, \subseteq)$  satisfying AA1, AA2, AA3 is an e-algebra. According to the definition of an e-algebra it has to be proved is that multiplication by scalars is inclusion isotone. Thus we have to show  $a \subseteq b \implies \gamma * a \subseteq \gamma * b$ , for  $a, b \in \mathbb{S}$ ,  $\gamma \in \mathbb{R}$ . If  $a, b \in \mathbb{S}_+$ , using A3.3., we have  $(\gamma * 1_+) \times a \subseteq (\gamma * 1_+) \times b$ , which using AA3. implies  $\gamma * (1_+ \times a) \subseteq \gamma * (1_+ \times b)$ , that is  $\gamma * a \subseteq \gamma * b$ . If  $a, b \in \mathbb{S}_-$  we substitute above  $1_+$  by  $1_-$ . In the case  $a \in \mathbb{S}_-$ ,  $b \in \mathbb{S}_+$ , that is  $a \subseteq 0 \subseteq b$ , using the above method we can show the inclusions  $\gamma * a \subseteq 0$  and  $0 \subseteq \gamma * b$ , which imply  $\gamma * a \subseteq \gamma * b$ .  $\square$

**Theorem 7.** *The system  $\mathbb{R}_{ED} = (\mathbb{R}, +, \times, \mathbb{R}_D, *, \subseteq)$  with multiplication “ $\times$ ” defined by (9) and multiplication by scalars  $\gamma * a = |\gamma|a$ , is an e-algebra.*

Proof. We have to prove that for any scalars  $\alpha, \beta \in \mathbb{R}_D$  and any  $b, c \in \mathbb{R}$  the following four relations hold: i)  $\alpha * (\beta * c) = (\alpha\beta) * c$ ; ii)  $\alpha * (b + c) = \alpha * b + \alpha * c$ ; iii)  $1 * c = c$ ; iv)  $(\alpha + \beta) * c = \alpha * c + \beta * c$  if  $\alpha\beta \geq 0$ . Using  $\gamma * a = |\gamma|a$  verification is straightforward.  $\square$

**Remark.** In an e-algebra  $\mathbb{R}_{ED} = (\mathbb{R}, +, \times, \mathbb{R}_D, *, \subseteq)$  the two elements  $\gamma$  and  $a$  in a product by scalar  $\gamma * a$  (see the formulation of Theorem 5) are real numbers, but  $\gamma$  is from  $\mathbb{R}_D$  and  $a$  is from  $\mathbb{R}_{ED}$ . To avoid specifying which is the real scalar and which is the element of the e-ring, we stipulate to always place the real scalar (that is the element from  $\mathbb{R}_D$ ) as a first multiplier (on the left) and the element from  $\mathbb{R}_E$  as a second one (on the right). Thus in  $\mathbb{R}_{ED}$  we may write, e. g. an expression of the form  $a * b + b * c$ ; here it is understood that  $a \in \mathbb{R}_D$  and  $b, c, d \in \mathbb{R}_E$ . If we want to express the products in terms of the linear multiplication “ $\cdot$ ”, we have to make use of the formula:  $a * b = |a| \cdot b$  and formula (9).

Clearly, an e-algebra  $\mathbb{R}_{ED} = (\mathbb{R}, +, \times, \mathbb{R}_D, *, \subseteq)$  is a triple of the form  $(\mathbb{R}_E, \mathbb{R}_D, *)$ , such that multiplication in  $\mathbb{R}_E$  and multiplication by scalars in  $\mathbb{R}_D$  are consistent. In the next section we shall consider the direct sum of the spaces  $\mathbb{R}_D$  and  $\mathbb{R}_{ED}$ . The space  $\mathbb{R}_D \oplus \mathbb{R}_{ED}$  abstractly summarizes certain features of the space of generalized (not necessarily centred) intervals.

#### 4.1 Spaces of Approximate Numbers (Intervals)

Consider the direct sum  $\mathbb{R}_D \oplus \mathbb{R}_{ED}$ . The element  $a = (a'; a'') \in \mathbb{R}_D \oplus \mathbb{R}_{ED}$ , is called an *approximate number* [36] or an *interval*. Clearly,  $(a'; 0) \in \mathbb{R}_D$  and  $(0; a'') \in \mathbb{R}_{ED}$ . Addition of approximate numbers is:

$$a + b = (a'; a'') + (b'; b'') = (a' + b'; a'' + b'').$$

The opposite (inverse additive) of  $a = (a'; a'')$  is:

$$\text{opp}(a'; a'') = \neg(a'; a'')_- = (-a'; a''_-) = (-a'; -a'').$$

In  $\mathbb{R}_D \oplus \mathbb{R}_{ED}$  we have multiplication by scalars (from  $\mathbb{R}_D$ ):

$$\gamma * (a'; a'') = (\gamma * a'; \gamma * a'') = (\gamma \cdot a'; \gamma * a'') = (\gamma a'; |\gamma| a''), \quad \gamma \in \mathbb{R}_D. \quad (50)$$

Note that  $(-1) * (a'; a'') = \neg(a'; a'') = (-a'; a'') \neq \text{opp}(a'; a'')$ .

**Theorem 8.** *The system  $(\mathbb{R}_D \oplus \mathbb{R}_{ED}, +, \mathbb{R}_D, *)$  is a quasivector space.*

Proof. We know that  $\mathbb{R}_D$  is a vector space and therefore is quasivector one. Also  $(\mathbb{R}_{ED}, +, \mathbb{R}_D, *)$  is a quasivector space. The direct sum of two quasivector spaces  $(\mathbb{R}_D \oplus \mathbb{R}_{ED}, +, \mathbb{R}_D, *)$  is also a quasivector space.  $\square$

We have  $(-1) * (b'; b'') = (-b'; b'')$ . We shall further denote  $(-1) * a$  by  $\neg a$  and call the operation  $\neg a$  negation  $\neg(a'; a'') = (-a'; a'')$ .

We denote the composition of “opp” and “neg” in  $\mathbb{R}_D \oplus \mathbb{R}_{ED}$  as *conjugation (dual)*, symbolically  $a_- = \text{opp}(\neg a) = \neg(\text{opp}(a))$ . Note that in  $\mathbb{R}_{ED}$  negation is identity,  $\neg(0; a'') = (0; a'')$ , and we have  $a_- = \text{opp}(a)$  for  $a \in \mathbb{R}_{ED}$ , that is  $(0; a'')_- = \text{opp}(0; a'') = (0; -a''_-) = (0; -a'')$ .

On the other side, in  $\mathbb{R}_D$  we have  $(a'; 0)_- = (a'; 0)$ , so that conjugation and identity coincide in  $\mathbb{R}_D$ . In general we have:

$$a_- = (a'; a'')_- = (-1) * \text{opp}(a) = (a'; a''_-) = (a'; -a'').$$

#### 4.1.1 Multiplication

We define *multiplication* in  $\mathbb{R}_D \oplus \mathbb{R}_{ED}$  by:

$$\begin{aligned} (a'; a'') \times (b'; b'') &= (a' \cdot b'; a' * b'' + b' * a'' + a'' \times b'') \\ &= (a'b'; |a'|b'' + |b'|a'' + a'' \times b''), \end{aligned} \quad (51)$$

using the same notation “ $\times$ ” as in  $\mathbb{R}_{ED}$ . From the special cases:

$$\begin{aligned} (a'; 0) \times (b'; 0) &= (a'b'; 0), \\ (0; a'') \times (0; b'') &= (0; a'' \times b''), \end{aligned}$$

we conclude that (51) extends the multiplications from  $\mathbb{R}_D$  and  $\mathbb{R}_{ED}$  into their direct sum  $\mathbb{R}_D \oplus \mathbb{R}_{ED}$ .

We also have

$$(a'; 0) \times (b'; b'') = (a'b'; a' * b'') = (a'b'; |a'|b''), \quad (52)$$

showing that a multiplier of the form  $(a'; 0)$  acts like a scalar in multiplication by scalars (50). Hence elements of the form  $(a'; 0)$  can be identified

with scalars  $a' \in \mathbb{R}_D$ . In particular, we have  $(-1; 0) \times (b'; b'') = (-b'; b'')$ .

**Commutativity.** It is immediately seen that multiplication (51) in  $\mathbb{R}_D \oplus \mathbb{R}_{ED}$  is commutative.

**Associativity.** Multiplication (51) in  $\mathbb{R}_D \oplus \mathbb{R}_{ED}$  is not associative, as the following result shows.

**Proposition 13.** *The relation  $a \times (b \times c) = (a \times b) \times c$  holds true, if  $a, b, c$  are all proper, or are all improper.*

Proof. Using (11) we have

$$\begin{aligned}
a \times (b \times c) &= (a'; a'') \times ((b'; b'') \times (c'; c'')) = (a'; a'') \times (b'c'; b' * c'' + c' * b'' + b'' \times c'') \\
&= (a'(b'c'); a' * (b' * c'' + c' * b'' + b'' \times c'')) + (b'c') * a'' \\
&+ a'' \times (b' * c'' + c' * b'' + b'' \times c'') \\
&= (a'b'c'; (a'b') * c'' + (a'c') * b'' + a' * (b'' \times c'')) + (b'c') * a'' \\
&+ b' * (c'' \times a''_{\tau(b \times c)\tau(c)}) + c' * (b'' \times a''_{\tau(b \times c)\tau(b)}) + b'' \times c'' \times a''_{\tau(b \times c)\tau(b)\tau(c)}.
\end{aligned}$$

On the other side,

$$\begin{aligned}
(a \times b) \times c &= ((a'; a'') \times (b'; b'')) \times (c'; c'') = (a'b'; a' * b'' + b' * a'' + a'' \times b'') \times (c'; c'') \\
&= ((a'b')c'; (a'b') * c'' + c' * (a' * b'' + b' * a'' + a'' \times b'')) \\
&+ (a' * b'' + b' * a'' + a'' \times b'') \times c'' \\
&= (a'b'c'; (a'b') * c'' + (c'a') * b'' + (c'b') * a'' + c' * (a'' \times b'')) \\
&+ a' * (b'' \times c''_{\tau(a \times b)\tau(b)}) + b' * (a'' \times c''_{\tau(a \times b)\tau(a)}) + a'' \times b'' \times c''_{\tau(a \times b)\tau(a)\tau(b)}.
\end{aligned}$$

In the case  $\sigma(a'') = \sigma(b'') = \sigma(c'') = +$  both sides clearly coincide; similarly we check the case  $\sigma(a'') = \sigma(b'') = \sigma(c'') = -$ . If  $\sigma(a'') = \sigma(b'') = \sigma(c'')$  does not hold then relation  $a \times (b \times c) = (a \times b) \times c$  is generally violated.  $\square$

**Identity.** It is immediately seen that  $(1; 0)$  is an identity of  $(\mathbb{R}_D \oplus \mathbb{R}_{ED}, \times)$ :

$$(a'; a'') \times (1; 0) = (a'; a'').$$

**Reciprocal.** The solution  $x$  of the equation  $a \times x = 1$  for  $a' \neq 0$  is  $x = 1/a = a^{-1} = (1/a'; -a''/|a'|^2)$ . We say that  $a^{-1}$  is the reciprocal of  $a$ .

Indeed, we have:

$$\begin{aligned}
a \times a^{-1} &= (a'; a'') \times (1/a'; -a''/|a'|^2) \\
&= (1; (1/|a'|) * a'' + a' * (-a''/|a'|^2) + a'' \times (-a''/|a'|^2)) \\
&= (1; a''/|a'| + |a'|(-a''/|a'|^2) + 0) = (1; 0).
\end{aligned}$$

We thus proved the following

**Proposition 14.** *The system  $(\mathbb{R}_D \oplus \mathbb{R}_{ED}, \times)$  is a commutative groupoid with identity.*

**Proposition 15.** *The operations  $\times$  and  $*$  are consistent, that is*

$$(\gamma * (a'; a'')) \times (b'; b'') = \gamma * ((a'; a'') \times (b'; b'')).$$

Proof. Both sides can be written

$$\begin{aligned}
(\gamma * (a'; a'')) \times (b'; b'') &= (\gamma a'; |\gamma| a'') \times (b'; b'') \\
&= (\gamma a' b'; |\gamma a'| b'' + |\gamma b'| a'' + |\gamma| (a'' \times b'')), \\
\gamma * ((a'; a'') \times (b'; b'')) &= \gamma * (a' b'; |a'| b'' + |b'| a'' + (a'' \times b'')) \\
&= (\gamma a' b'; |\gamma a'| b'' + |\gamma b'| a'' + |\gamma| (a'' \times b'')),
\end{aligned}$$

which proves the property.  $\square$

We have:

$$(0; a'') \times (b'; b'') = (0; b' * a'' + a'' \times b'') = (0; |b'| a'' + a'' \times b''), \quad (53)$$

showing that  $\mathbb{R}_{ED}$  is an “ideal” of  $\mathbb{R}_D \oplus \mathbb{R}_{ED}$ . In particular,  $(a'; 0) \times (0; b'') = (0; a' * b'') = (0; |a'| b'')$ .

We note that formula (51) is a familiar construction for endowing the e-algebra  $\mathbb{R}_{ED} = (\mathbb{R}, +, \times, \mathbb{R}_D, *, \subseteq)$  with an identity. That is, we embed a ringoid without identity into a direct sum of this ringoid and a ring with identity, cf. [1], [3] (p. 337), [34]. In our case as a ring with identity we use the familiar ring  $\mathbb{R}_D$ , arriving thus to the direct sum  $\mathbb{R}_D \oplus \mathbb{R}_{ED}$ .

**Distributivity.** We shall next investigate distributivity in  $\mathbb{R}_D \oplus \mathbb{R}_{ED}$ . Using that

$$\begin{aligned}
\text{opp}((a'; a'')) &= (-a'; -a''), \\
(a'; a'')_- &= (a'; -a''), \\
\neg(a'; a'') &= (-a'; a''),
\end{aligned}$$



it is easy to check:  $(a \times b)_- = \text{opp}(a) \times \text{opp}(b)$ ,  $(a \times b) = (-a) \times (-b)$ . Multiplication is distributive with respect to conjugation:  $(a \times b)_- = (a_-) \times (b_-)$ .

Using (52) and (53) we have

$$\begin{aligned} a \times b &= (a'b'; a' * b'' + b' * a'' + a'' \times b'') \\ &= (a'b'; a' * b'') + (0; b' * a'' + a'' \times b'') \\ &= (a'; 0) \times (b'; b'') + (0; a'') \times (b'; b''), \\ &= (a'; 0) \times b + (0; a'') \times b. \end{aligned}$$

The relation  $a \times b = ((a'; 0) + (0; a'')) \times b$  can be written  $((a'; 0) + (0; a'')) \times c = (a'; 0) \times c + (0; a'') \times c$ , or, the following distributive relation holds:

$$((a'; 0) + (0; b'')) \times c = (a'; 0) \times c + (0; b'') \times c. \quad (54)$$

We next want to extend (54) for the case when the left-hand side of the equality has the more general form  $((a'; a'') + (b'; b'')) \times c$ . In what follows we shall write  $\sigma(a'') = \tau(a'')$ , there will be no confusion.

We define  $\sigma(a)$  for  $a \in \mathbb{R}_D \oplus \mathbb{R}_{ED}$  as a pair of signs:

$$\sigma(a) = \sigma(a'; a'') = (\sigma(a'), \sigma(a'')) = (\sigma(a'), \tau(a'')).$$

Further we define:

$$\begin{aligned} a \times b_{\sigma(a)} &= ((a'; a'')) \times b_{(\sigma(a'), \sigma(a''))} \\ &= ((a'; 0) + (0; a'')) \times b_{(\sigma(a'), \sigma(a''))} \\ &= (a'; 0) \times b_{\sigma(a')} + (0; a'') \times b_{\sigma(a'')}. \end{aligned} \quad (55)$$

Thus we have:

$$\begin{aligned} a \times b_{\sigma(a)} &= (a'; 0) \times b_{\sigma(a')} + (0; a'') \times b_{\sigma(a'')} \\ &= (a'; 0) \times (b'; b''_{\sigma(a')}) + (0; a'') \times (b'; b''_{\sigma(a'')}) \\ &= (a'b'; a' * b''_{\sigma(a')}) + (0; b' * a'' + a'' \times b''_{\sigma(a'')}) \\ &= (a'b'; a' * b''_{\sigma(a')} + b' * a'' + a'' \times b''_{\sigma(a'')}). \end{aligned} \quad (56)$$

Using notation (55) we formulate the following law in  $\mathbb{R}_D \oplus \mathbb{R}_{ED}$ :

**Theorem 9.** For  $a, b, c \in \mathbb{R}_D \oplus \mathbb{R}_{ED}$ :

$$(a + b) \times c_{\sigma(a+b)} = a \times c_{\sigma(a)} + b \times c_{\sigma(b)}. \quad (57)$$

Proof. Using (55), (55) and the quasidistributive laws in a e-ring (45) and in a symmetric q-linear space (39), we obtain:

$$\begin{aligned}
a \times c_{\sigma(a)} &+ b \times c_{\sigma(b)} \\
&= (a'c'; a' * c''_{\sigma(a')} + c' * a'' + a'' \times c''_{\sigma(a'')}) \\
&+ (b'c'; b' * c''_{\sigma(b')} + c' * b'' + b'' \times c''_{\sigma(b'')}) \\
&= ((a' + b')c'; (a' + b') * c''_{\sigma(a'+b')}) \\
&+ c' * (a'' + b'') + (a'' + b'') \times c''_{\sigma(a''+b'')} \\
&= (a + b) \times c_{\sigma(a+b)}.
\end{aligned}$$

□

As special cases of the distributivity relation (57) we obtain:

$$((a'; 0) + (0; b'')) \times (c'; 0) = (a'c'; c' * b'') = (a'c'; |c'|b'')$$

$$((a'; 0) + (0; a'')) \times (0; c'') = (0; a' * c'' + b'' \times c'') = (0; |a'|b'' + b'' \times c'').$$

**Inclusion isotonicity of multiplication.** The relation  $\subseteq$  is extended from  $\mathbb{R}_E$  to  $\mathbb{R}_D \oplus \mathbb{R}_{ED}$  as follows:

$$a \subseteq b \iff |b' - a'| \leq b'' - a''. \quad (58)$$

Note that the condition  $|b' - a'| \leq b'' - a''$  means that there exists a  $\gamma \in \mathbb{R}$  such that  $0 \leq \gamma = b'' - a''$  and  $|b' - a'| \leq \gamma$ .

The inclusion  $0 \subseteq a = (a'; a'')$  is equivalent to  $|a'| \leq a''$  and the inclusion  $b = (b'; b'') \subseteq 0$  is equivalent to  $|b'| \leq -b''$ , that is to  $b'' \subseteq 0$  ( $b'' \leq 0$ ) and  $|b'| \leq -b''$ .

If  $b = (b'; b'')$  is such that  $b'' \geq 0$  and  $a = (a'; 0)$  then  $a \subseteq b$  means  $|b' - a'| \leq b''$ , that is  $-b'' \leq b' - a' \leq b''$ .

Inclusion isotonicity of addition:  $a \subseteq b \iff a + c \subseteq b + c$  is an immediate consequence of (58). We now prove inclusion isotonicity of interval multiplication.

**Proposition 16.** *If  $a, b, c \in \mathbb{R}_D \oplus \mathbb{R}_{ED}$ , then*

$$a \subseteq b \implies a \times c \subseteq b \times c. \quad (59)$$

Proof. Relation  $a \times c \subseteq b \times c$  is, according to (58):

$$\begin{aligned}
|b'c' - a'c'| &\leq b' * c'' + c' * b'' + b'' \times c'' - (a' * c'' + c' * a'' + a'' \times c'') \\
&= |b'|c'' + |c'|b'' + b'' \times c'' - (|a'|c'' + |c'|a'' + a'' \times c'') \\
&= (|b'|c'' - |a'|c'') + (|c'|b'' - |c'|a'') + (b'' \times c'' - a'' \times c'') \\
&= (||b'| - |a' ||)c''_{\sigma(|b'| - |a'|)} + |c'|(b'' - a'') + (b'' \times c'' + a'' \times c''_-) \\
&= (||b'| - |a' ||)c''_{\sigma(|b'| - |a'|)} + |c'|(b'' - a'') + (b'' \times c'' + a''_- \times c''_-).
\end{aligned}$$

**Theorem 10.**  $(\mathbb{R}_D \oplus \mathbb{R}_{ED}, +, \times, \subseteq)$  is an commutative e-ring with identity.

#### 4.1.2 Division

For  $a \in \mathbb{R}_D \oplus \mathbb{R}_{ED}$  denote  $\tau(a'; a'') = \sigma(a'') = \tau(a'')$ .

We defined the reciprocal  $1 \not\prec a = a^{-1}$  of  $a$  as the solution  $x$  of the equation  $a \times x = 1$  for  $a' \neq 0$ . We have  $1 \not\prec a = a^{-1} = (1/a'; -a''/|a'|^2)$ .

Let us examine the solution  $x$  of the equation  $a \times x = b$ . Denoting  $a = (a'; a'')$ ,  $b = (b'; b'')$ ,  $x = (x'; x'')$ , we have:

$$\begin{aligned}
(a'; a'') \times (x'; x'') &= (a'x'; a' * x'' + x' * a'' + a'' \times x'') \\
&= (a'x'; |a'|x'' + |x'|a'' + a'' \times x'').
\end{aligned}$$

Thus  $a \times x = b$  can be written component-wise:  $(a'x'; |a'|x'' + |x'|a'' + a'' \times x'') = (b'; b'')$ .

Assuming  $a' \neq 0$ , the latter gives  $x' = b'/a'$  and the equation for  $x''$ :  $|a'|x'' + |b'/a'|a'' + a'' \times x'' = b''$ , which can be written as:

$$|a'|x'' + (a'' \times x'') = b'' - \frac{|b'|}{|a'|}a''. \quad (60)$$

or, assuming  $b' \neq 0$ ,

$$x'' + \frac{1}{|a'|}(a'' \times x'') = \frac{|b'|}{|a'|} \left( \frac{b''}{|b'|} - \frac{a''}{|a'|} \right). \quad (61)$$

We look for possible solutions for  $x''$  of (61). Note that the sign of the right hand-side

$$r = \frac{|b'|}{|a'|} \left( \frac{b''}{|b'|} - \frac{a''}{|a'|} \right)$$

of (61) depends on the ratio of  $\kappa(a)$  and  $\kappa(b)$ , where  $\kappa(a) = a''/|a'|$  is introduced and studied in [10]. Note also that, according to (9) the product  $a'' \times x''$

is either equal to 0 (if  $\sigma(x'') \neq \sigma(a'')$ ) or is equal to  $|a''|x''$  (if  $\sigma(x'') = \sigma(a'')$ ); hence the sign of the left hand-side of (61) is equal to the sign of  $x''$ .

We now consider different cases.

1. Assume  $\sigma(a'') = -\sigma(b'')$ . Then  $\sigma(r) = \sigma(b'') = -\sigma(a'')$ . As the left hand-side of (61) has the sign of  $x''$ ,  $\sigma(x'') = \sigma(r)$ , we see that  $x''$  and  $a''$  have different signs and hence  $a'' \times x'' = 0$ . Thus (61) becomes

$$x'' = \frac{|b'|}{|a'|} \left( \frac{b''}{|b'|} - \frac{a''}{|a'|} \right) = \left( \frac{b''}{|a'|} - \frac{a''}{|a'|^2} \right), \quad (62)$$

so that in this case:

$$x = b \not\! \times a = (b'/a'; \frac{|b'|}{|a'|} \left( \frac{b''}{|b'|} - \frac{a''}{|a'|} \right)). \quad (63)$$

2. Assume  $\sigma(a'') = \sigma(b'')$ . Then

$$\sigma(x'') = \begin{cases} \sigma(b'') = \sigma(a''), & \text{if } |\kappa(b)| \geq |\kappa(a)|, \\ -\sigma(b'') = -\sigma(a''), & \text{if } |\kappa(b)| < |\kappa(a)|. \end{cases}$$

In the subcase  $|\kappa(b)| \geq |\kappa(a)|$ , we have  $\sigma(x'') = \sigma(a'')$ , resp., by (??),  $a'' \times x'' = |a''|x''$ , so that (61) becomes

$$x'' = \frac{|b'|}{|a'| + |a''|} \left( \frac{b''}{|b'|} - \frac{a''}{|a'|} \right) = \frac{1}{|a'| + |a''|} \left( b'' - \frac{|b'|}{|a'|} a'' \right), \quad (64)$$

so that

$$x = b \not\! \times a = (b'/a'; \frac{|b'|}{|a'| + |a''|} \left( \frac{b''}{|b'|} - \frac{a''}{|a'|} \right)). \quad (65)$$

In the subcase  $|\kappa(b)| < |\kappa(a)|$ , we have  $\sigma(x'') = -\sigma(a'')$ , resp.,  $x'' \times a'' = 0$ , so that (61) becomes

$$x'' = \frac{|b'|}{|a'|} \left( \frac{b''}{|b'|} - \frac{a''}{|a'|} \right), \quad (66)$$

resp.

$$x = b \not\! \times a = (b'/a'; \frac{|b'|}{|a'|} \left( \frac{b''}{|b'|} - \frac{a''}{|a'|} \right)). \quad (67)$$

It is interesting to note that

$$\begin{aligned} x = b \not\! \times a &= b \times (1 \not\! \times a) = (b'; b'') \times (1/a'; -a''/|a'|^2) \\ &= (b'/a'; -(|b'|/|a'|^2)a'' + (1/|a'|)b''). \end{aligned}$$

The above calculations can be summarized in the following

**Proposition 17.** *The solution of  $a \times x = b$  for the case  $a' \neq 0$  is  $x = (b'/a'; x'')$ , where*

$$x'' = \begin{cases} \frac{|b'|}{|a'|+|a''|} \left( \frac{b''}{|b'|} - \frac{a''}{|a'|} \right), & \text{if } \sigma(a'') = \sigma(b'') \text{ and } |\kappa(b)| \geq |\kappa(a)|, \\ x'' = \frac{|b'|}{|a'|} \left( \frac{b''}{|b'|} - \frac{a''}{|a'|} \right) = b \times (1 \not\approx a), & \text{otherwise.} \end{cases}$$

Proof. It is easy to check that for  $a' \neq 0$  the approximate number  $x = (b'/a'; x'')$ , where  $x''$  is given by the above formula, is a solution of  $a \times x = 1$ , resp. of  $a \times x = b$ .  $\square$

Consider now the case  $a' = 0$ . In this case we must have  $b' = 0$  as well. We also assume  $a'' \neq 0$ , as the case  $a'' = 0$  is not interesting. We thus have:

$$(0, a'') \times (x'; x'') = (0; x' * a'' + a'' \times x'') = (0; b''),$$

that is we must solve  $|x'|a'' + a'' \times x'' = b''$ . If  $b'' = 0$ , then every  $x''$  with  $\tau(x'') = -\tau(a'')$  is a solution for  $(0; a'') \times (0; x'') = (0; 0)$  due to  $a'' \times x'' = 0$ . Consider the case  $b'' \neq 0$ . Assume first that  $\tau(a'') = \tau(b'')$ . In the subcase  $|x'|a'' \leq b''$ , that is  $|x'| \leq b''/a''$  we have  $a'' \times x'' = b'' - |x'|a''$  showing that  $a'' \times x''$  has the sign of  $b''$  (of  $a''$ ). Hence we can write  $|a''|x'' = b'' - |x'|a''$ , that is:

$$x'' = (b'' - |x'|a'')/|a''|,$$

$$\text{e. g. } x'' = b''/|a''| \quad (x' = 0),$$

so that the solution is  $x = (0; b''/|a''|)$ .

Indeed, we have  $(0; a'') \times (0; b''/|a''|) = (0; b'')$ . More generally, we have:

$$x' = \varepsilon(b''/a''), \quad \varepsilon \in [-1, 1], \quad x'' = (b'' - |x'|a'')/|a''|.$$

It is easy to see that there is no solution in the rest of the cases.

We used the following property:

$$\text{If } a \times b \neq 0, \text{ then } a \times b = |a|b = |b|a, \text{ and } \tau(a \times b) = \tau(a) = \tau(b).$$

## 5 Concluding Remarks

We have shown that errors, resp. symmetric intervals, possess algebraic properties somewhat different from the properties of the real numbers. Our investigations show the importance of the midpoint-radius representation of intervals, already noticed in [33], see also [14]. Similar representation has been recently announced in an advanced software for interval computations [32], [31].

The theory of e-rings can be used to develop a general abstract algebraic theory of intervals. We wish to note that the quasidistributive law (45) is a special case of the quasidistributive relations for generalized intervals obtained in [26].

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Appendix 1.

**The hierarchy of interval spaces:**

1. *Familiar sets and algebraic systems used to define interval systems:*

$\mathbb{R}$                       The set of reals

$\mathbb{R}_D = (\mathbb{R}, +, \cdot, \leq)$                       The ordered real field

$\mathbb{R}^n$                       The set of real  $n$ -tuples

$\mathbb{R}_D^n = (\mathbb{R}^n, +, \mathbb{R}_D, \cdot, \leq)$     The ordered ( $n$ -dim.) linear space ( $\mathbb{R}_D^1 = \mathbb{R}_D$ )

2. *Quasivector spaces (with group structure)*

$\mathbb{R}_D^n = (\mathbb{R}^n, +, \mathbb{R}_D, *, \leq)$     The ordered ( $n$ -dim.) centred quasivector space; former:  $\mathbb{R}_E^n = (\mathbb{R}^n, +, \mathbb{R}_D, *, \leq)$  or  $\mathbb{R}_E^n = (\mathbb{R}^n, +, \mathbb{R}_D, *, \subseteq)$

$\mathbb{R}_{ED}^n = (\mathbb{R}^n, +, \mathbb{R}_D, \cdot, *, \leq)$     The ordered ( $n$ -dim.) (centred quasi)vector space (extended, mixed)

$\mathbb{R}_E^1 = (\mathbb{R}, +, \mathbb{R}_D, *, \subseteq)$     The one-dimensional case     $\mathbb{R}_E^1 \neq \mathbb{R}_E$  ?

$(\mathbb{R}_D^n \oplus \mathbb{R}_{ED}^n, +, \mathbb{R}_D, *)$     The quasivector space (noncentred)

$\mathbb{R}_{DED}^n = \mathbb{R}_D^n \oplus \mathbb{R}_{ED}^n = (\mathbb{R}^{2n}, +, \mathbb{R}_D, *)$  The quasivector space (noncentred)

$\mathbb{I}^n = (\mathbb{R}_E^n \oplus \mathbb{R}_D^n, +, \mathbb{R}_D, *, \subseteq)$  The *ordered* quasivector space

$\mathbb{I}^1 = (\mathbb{R}_E^1 \oplus \mathbb{R}_D, +, \mathbb{R}_D, *, \subseteq)$  The one-dim. case

3. *E-ring*:

$\mathbb{R}_E = (\mathbb{R}, +, \times, \subseteq)$  The e-ring (e-field)  $\tau$ -operation

4. *Isotone systems (e-spaces, i-spaces) — always ordered*:

$\mathbb{R}_{ED} = (\mathbb{R}, +, \times, \mathbb{R}_D, *, \subseteq) = (\mathbb{R}_E, \mathbb{R}_D, *)$  The e-algebra

$\mathbb{I} = (\mathbb{R}_D \oplus \mathbb{R}_{ED}, +, \times, \subseteq)$

$\mathbb{I} = (\mathbb{R}_D \oplus \mathbb{R}_{ED}, +, \times, \mathbb{R}_D, *, \subseteq)$  The i-algebra

$$\mathbb{I}^n = (\mathbb{R}_D \oplus \mathbb{R}_{ED})^n = \mathbb{R}_D^n \oplus \mathbb{R}_{ED}^n$$

Appendix 2.

Using the operation hyperbolic multiplication of generalized intervals in end-point form  $a = [a^-, a^+], b = [b^-, b^+]$

$$a \cdot b = [a^- b^-, a^+ b^+]$$

E. Kaucher gives the following formula for multiplication of generalized intervals [8]:

$$p = \begin{cases} a_{\sigma(b)} b_{\sigma(a)}, & \text{if } a, b \in \mathcal{S}_r \cup \mathcal{S}_l \\ \sigma(b') |b| * a, & \text{if } \text{Cond}(a, b); \\ \sigma(a') |a| * b, & \text{if } \text{Cond}(b, a); \\ (0; 0), & \text{if } a \in \mathcal{T}_f, b \in \mathcal{T}_p \text{ or } b \in \mathcal{T}_f, a \in \mathcal{T}_p, \end{cases} \quad (68)$$

wherein

$$\begin{aligned} \mathcal{S}_r &= \{a \mid a^- \geq 0, a^+ \geq 0\}; \\ \mathcal{S}_l &= \{a \mid a^- \leq 0, a^+ \leq 0\}; \\ \mathcal{T}_f &= \{a \mid a^- \leq 0 \leq a^+\}; \\ \mathcal{T}_p &= \{a \mid a^+ \leq 0 \leq a^-\}. \end{aligned}$$