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The k-calculus Approach and the Generalized Lorentz Transformations

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1 Introduction

In [1] J. Winnie develops the consequences of the Special Theory of Relativity (STR) when no assumptions are made concerning the one-way velocity of light. He weakens the Second Postulate of STR to the so-called Round-trip Light Principle (RTLP), which is consistent with the Reichenbach-Grünbaum thesis of the conventionality of simultaneity. The RTLP requires the average round-trip speed of light to be c whereas the one-way speed of the light signals is allowed to depend on the conventional choice of a parameter ε.

Winnie's model [1] based on the Relativity Principle and on the RTLP we shall further refer to as ε-STR. As Winnie shows ε-STR is equivalent to STR as regards the results based on experiments with light signals propagating in a closed path (so-called two-way experiments). The quantities and relations which do not depend on the particular choice of ε we shall call (as Winnie does) synchrony-independent.

The purpose of this paper is twofold. First, we note that H. Bondi's definition of chronometric motion [2], as being based on an one-way thought experiment, might contain a conventional ingredient. In order to use this definition separately from STR, we restate the definition by basing it on a two-way experiment. Then the k-calculus approach [2, 3] is systematically exploited in ε-STR, mainly in order to discuss the synchrony-independence of certain quantities and results. Deriving the general Lorentz formulae we use an approach which might have some methodological advantages. Such an inductive approach is often used in modern mathematical modelling. It consists of i) proper choice of the general form of the mathematical relations which map the physical reality and ii) determination of the parameters in these relations on the basis of certain physical assumptions. Such an approach to Special Relativity can be found in the textbook [4]. As it is significant for this approach to distinguish the physical concepts from the mathematical ones, we use throughout the paper different types of notations (boldface upper case for the physical objects and italics for the mathematical concepts).

Second, we note the fact that the RTLP contains the assumption that the velocity of light does not depend on the velocity of the source. Our approach allows us to study easily the consequences of the assumption that such a dependence exists; let us note that the experiments of Michelson-Morly type does not reject this dependence and that the direct one-way experiments (in high vacuo) performed in the recent years might not be regarded as crucial [5, 6]. Among the hypotheses permitting the dependence of the velocity of light on the velocity of the source (in vacuo), the most popular is that of Ritz, which states that the velocity of light is equal to a constant as regards the source. Our approach to deriving the Lorenz transformations allows to see easily how the transformations look like when Winnie's formulation of the Relativity Principle and the Ritz hypothesis are taken together as physical basis.

2 The k-calculus Approach

Let **S** and **S**$'$ be two inertial frames of reference moving relatively in a standard position. We confine our attention throghout the paper to events on the x-axes of the frames involved. Denote by K and K' the corresponding two-dimensional coordinate systems. (The mathematical relations between K and K' will be discussed later on.) To each physical event **P** we associate two pairs of real numbers: (x, t) as regards K and (x', t') as regards K'; x shows the place of the event **P** on the x-axis of **S**, t shows the time by the clock $\mathbf{U}_x$ fixed at that place at rest in **S**, etc. The motion of a point **A** can be considered as a set of events; its geometrical image is then a point set W_A called world-line of **A**.

H. Bondi [2, 3] suggests an experiment with light signals for identifying a certain class of motions called by him chronometric. Consider two observers **O** and **O**$'$ moving on a straight line. By means of a radar **O** sends two signals in succession to **O**$'$; let T be the time interval between the instances of the dispatching of the signals (by $\mathbf{U}_O$). The observer **O**$'$ reads (by his clock $\mathbf{U}_{O'}$ a time interval T' between the instances of receiving the signals. It is also assumed that while this experiment, which we shall denote by **E**, takes place, **O** and **O**$'$ do not pass each other. Then the definition given by Bondi can be stated as follows:

Definition 1. Let **O** and **O**$'$ be in relative motion and let for any experiment **E** the intervals T and T' relate as

(1) $$T' = kT,$$

where k is constant in time independent of T. Then we say that **O**$'$ moves chronometrically with respect to **O** with coefficient of chronometry k (or **O**$'$ moves k-chronometrically with respect to **O**).

The experiment **E** assumes reading of time by two clocks in relative motion, which means that we have to know the way the clocks are synchronized. This concerns the hypothesis assumed about the one-way speed of light and eventually makes difficult to see whether k is synchrony-independent or not. That is why let us consider another thought experiment.

Let the observer **O** dispatch again two signals in succession to **O**′ and read between them time T (by his clock $\mathbf{U}_O$). The signals reflect from **O**′ and return to **O**. Let then T'' be the time interval (again by $\mathbf{U}_O$, which **O** reads between the instances of receiving the reflected signals. This experiment we shall denote by $\mathbf{E}^*$.

Definition 2. If for any experiment $\mathbf{E}^*$ between the time intervals T and T'' the relation

$$T'' = k^2 T \tag{2}$$

holds, where k is a constant in time, we shall say that **O**′ moves k-chronometrically with respect to **O**.

In the experiment $\mathbf{E}^*$ all the readings are made by one clock. Thus Definition 2 allows to determine (on principle) the chronometricity of a motion using one clock and also independently of any one-way hypothesis.

It is easy to see that the two definitions are equivalent in STR in the sense that either of them follows from the other. In Section 5 we show that both definitions are also equivalent in ε-STR which means that the coefficient k is a synchrony-independent quantity.

Let **O** be the origin of the frame **S** and let the point **O**′ move in the positive direction of **S** chronometrically with respect to **O** (according to Definition 2). Suppose that **O**′ approaches **O**, then passes "through" **O** (at some instant t_0 by $\mathbf{U}_O$) and with unchanged velocity recedes from **O**. Obviously, the chronometry coefficient measured before **O** and **O**′ pass is less than one and the coefficient measured after they pass is greater than one. The coefficient of chronometry measured by approaching we denote by k_a and by k_r when measured by receding.

Consider the case when **O**′ recedes from **O** chronometrically with coefficient k_r, after passing **O** at the instant t_0 (by $\mathbf{U}_O$). Assume that **O** sends the first signal to **O**′ at the instant t_0 and the second signal — at some instant $t_1 > t_0$. Then the first signal is received back again at t_0 (we assume that when **O** and **O**′ "coincide" the time for the to- and fro- signals is zero). Let the second signal return to **O** at the instant t_2 (by $\mathbf{U}_O$). When we substitute $T = t_1 - t_0$ and $T' = t_2 - t_0$ in (2) we get

$$t_2 - t_0 = k_r^2(t_1 - t_0). \tag{3}$$

Denote by $\vec{c}$ and $\overleftarrow{c}$ the velocities of the light signals moving in the positive and in the negative direction of **S**, respectively. Denote by (x, t) the event of reflection of the second signal from **O**′. We have then $t_1 = t - x/\vec{c}$, and $t_2 = t + x/\overleftarrow{c}$ which substituted in (3) gives $x = [(k_r^2 - 1)/(\overleftarrow{c}^{-1} + \vec{c}^{-1}\, k_r^2)](t - t_0)$. Thus the world-line of **O**′ with respect to K is a straight line; the constant factor in the above expression presents the velocity v_+ of **O**′ with respect to **S**.

Considering the case when **O**′ approaches **O** chronometrically with coefficient k_a (moving again in the positive direction of **S**) we obtain v_+ as a function of k_a. The following formula contains the two expressions for v_+

$$v_+ = \frac{1-k_a^2}{\overrightarrow{c}^{-1} + \overleftarrow{c}^{-1} k_a^2} = \frac{k_r^2 - 1}{\overleftarrow{c}^{-1} + \overrightarrow{c}^{-1} k_r^2}. \tag{4}$$

This formula relates the chronometry coefficients of one moving object measured before and after passing the observer. As far as k_a and k_r can be determined (at least on principle) by means of the experiment $\mathbf{E}^*$, it turns out that (4) presents a relation between the velocities $\overrightarrow{c}$ and $\overleftarrow{c}$ (any conventional choice of $\overrightarrow{c}$ determines uniquely $\overleftarrow{c}$).

For the case of a chronometric motion in the negative direction (again with chronometry coefficients k_a and k_r respectively) we get for the velocity v_- of the moving object similarly

$$v_- = \frac{1-k_a^2}{\overleftarrow{c}^{-1} + \overrightarrow{c}^{-1} k_a^2} = \frac{k_r^2 - 1}{\overrightarrow{c}^{-1} + \overleftarrow{c}^{-1} k_r^2}. \tag{5}$$

3 The Principle of Linearity and the Principle of Relativity

In this section we shall consider results based only on the Principle of Linearity and on the Principle of Relativity. The Principle of Linearity can be stated as follows.

Principle of Linearity. For any two inertial frames $\mathbf{S}$ and $\mathbf{S}'$ any point in inertial motion with respect to $\mathbf{S}$ is also in inertial motion with respect to $\mathbf{S}'$ and conversely.

We shall assume (as Bondi does) that the concept of chronometric motion in our kinematical model is identical to the concept of inertial motion in dynamics. As we have seen each chronometric motion is presented by a world-line of linear form with respect to K and K'. It is well known that the Principle of Linearity leads to linear transformation formulae between K and K' (which transform linear functions into linear functions). Let these formulae be of the form

$$\begin{aligned} x &= \alpha x' + \delta t' + a, \\ t &= \beta x' + \gamma t' + b, \end{aligned} \tag{6}$$

where $\alpha, \beta, \ldots$ are six constants, independent of the choice of $\mathbf{P}$.

The six constants in (6) determine the way K' lies in the plane with respect to K — the pairs (a, b), (α, β) and (δ, γ) are coordinates of the origin $\mathbf{O'}$ and the two unity vectors of K' with respect to K. Our next aim is to determine these constants.

We make the standard stipulation that when $\mathbf{O}$ and $\mathbf{O'}$ pass each other their clocks $\mathbf{U}_O$ and $\mathbf{U}_{O'}$, both read time zero. Thus the event $(0,0)$ with respect to K has coordinates $(0,0)$ with respect to K' and that implies $a = b = 0$.

The world-line of the origin $\mathbf{O}'$ of $\mathbf{S}'$ is $x = v_+ t$ with respect to K and $x' = 0$ with respect to K'. Applied to (6) this gives $\delta = v_+\gamma$. Thus the transformation formulae (6) become

$$(7) \qquad \begin{aligned} x &= \alpha x' + v_+\gamma t', \\ t &= \beta x' + \gamma t', \end{aligned} \qquad \text{or} \qquad \begin{aligned} &x' = (x - v_+ t)/(\alpha - v_+\beta), \\ &t' = (\alpha t - \beta x)/(\gamma(\alpha - v_+\beta)). \end{aligned}$$

Consider a set of consecutive events E_i, $i = 1, 2, \ldots$, at some fixed place x' in $\mathbf{S}'$, $E_i = (x', t'_i)$, $t'_i < t'_{i+1}$. Let E_i have coordinates (x_i, t_i) with respect to $\mathbf{S}$ (which can be obtained by (7)). We make the assumption that t_i also increase, i. e. $t_i < t_{i+1}$. This means that $t = \beta x' + \gamma t'$ is an increasing function of t' when x' is fixed. In other words, we assume that $\gamma > 0$.

Next we are going to see what is the restriction on the coefficients implied by the Relativity Principle. Winnie formulates the Relativity Principle in a form independent of the conventional choice of the one-way speed of light ([1], p. 230). We are going to show that the Relativity Principle (as formulated in [1]) bounds the coefficients in (7) to the relation $\gamma(\alpha - v_+\beta) = 1$. We shall note that Winnie's formulation of the Relativity Principle (called by him Principle of Equal Passage-Times) is substantially different from other formulations; the standard formulation (see, for example, [7]) implies the relation $\alpha(\alpha - v_+\beta) = 1$. The Relativity Principle is formulated by Winnie as follows:

Relativity Principle. Let $\mathbf{A}$ and $\mathbf{A}'$ be two arbitrary points on the x-axes of $\mathbf{S}$ and $\mathbf{S}'$, respectively. Let Δt be the time interval by the clock $\mathbf{U}_A$ (in $\mathbf{S}$) of the passage of a rod at rest in $\mathbf{S}'$ of length d (in $\mathbf{S}'$) past the point $\mathbf{A}$, and let $\Delta t'$ be the time interval by $\mathbf{U}_{A'}$ (in $\mathbf{S}'$) of the passage of a rod at rest in $\mathbf{S}$ of length d (in $\mathbf{S}$) past the point $\mathbf{A}'$. Then $\Delta t = \Delta t'$.

This formulation presents a simple assumption about the relation between the space and time units in the two inertial frames involved.

Let us take for $\mathbf{A}$ and $\mathbf{A}'$ the origins $\mathbf{O}$ and $\mathbf{O}'$ of the frames. Let $\mathbf{B}$ be a point at rest in $\mathbf{S}$ at a distance d from $\mathbf{O}$ and let $\Delta t'$ be the time interval by $\mathbf{U}_{O'}$ of the passage of the rod $\mathbf{OB}$ past $\mathbf{O}'$. Let $\mathbf{B}'$ be a point at rest in $\mathbf{S}'$ at a distance d in the negative direction from $\mathbf{O}'$ and let Δt be the time interval by $\mathbf{U}_{O'}$ of the passage of the rod $\mathbf{O}'\mathbf{B}'$ past $\mathbf{O}$. The world-line W_B of $\mathbf{B}$ is $x = d$ with respect to K or according to (7) $\alpha x' + v_+\gamma t' = d$ with respect to K'. Substituting here $x' = 0$ we get the time $\Delta t' = d/(v_+\gamma)$ by $\mathbf{U}_{O'}$ for the passage of $\mathbf{OB}$ past $\mathbf{O}'$. Similarly $\mathbf{B}'$ has world-line $x' = -d$ with respect to K' or $x - v_+ t = -d(\alpha - v_+\beta)$ with respect to K. Substituting $x = 0$ we obtain the time-interval $\Delta t = d(\alpha - v_+\gamma)/v_+$ by the clock U_O for the passage of $\mathbf{O}'\mathbf{B}'$ past $\mathbf{O}$. The comparison of Δt and $\Delta t'$ gives the following algebraical expression of the Relativity Principle:

$$(8) \qquad \gamma(\alpha - v_+\beta) = 1.$$

Thus the transformation formulae obtain the form

$$(9)\qquad \begin{aligned} x &= \alpha x' + v_+\gamma t', \\ t &= \beta x' + \gamma t', \end{aligned} \qquad \text{or} \qquad \begin{aligned} x' &= \gamma(x - v_+ t), \\ t' &= \alpha t - \beta x, \end{aligned}$$

wherein α, β and γ are related by means of (8).

Similarly, if $\mathbf{O}'$ moves in the negative direction of the x-axes of $\mathbf{S}$ with velocity v_-, for the transformation formulae we get

$$(10)\qquad \begin{aligned} x &= \overline{\alpha} x' - v_-\overline{\gamma} t', \\ t &= \overline{\beta} x' - \overline{\gamma} t', \end{aligned}$$

wherein $\overline{\alpha}, \overline{\beta}$ and $\overline{\gamma}$ are related by $\overline{\gamma}(\overline{\alpha} + v_-\overline{\beta}) = 1$.

Using (9) and (10) we shall derive now a general formula for the so-called round-trip time dilation effect (cf. [1], p. 96). This formula will be independent of any of the one-way velocity of light assumptions which are to be made later on.

Let $\mathbf{A}$ recede from $\mathbf{O}$ in the positive direction of $\mathbf{S}$ with constant velocity v_+ and at some distance s from $\mathbf{O}$ pass the point $\mathbf{B}$ which approaches $\mathbf{O}$ moving in the negative direction with constant velocity v_-. The clocks U_A and U_O read time zero as they pass each other. When $\mathbf{A}$ and $\mathbf{B}$ pass (at event M — see Fig. 1) the clock $\mathbf{U}_A$ reads time $t' = s/(v_+\gamma)$. Suppose the time by $\mathbf{U}_B$ is pre-set so that as $\mathbf{A}$ passes $\mathbf{B}$, $\mathbf{U}_B$ reads the same time as $\mathbf{U}_A$. We shall see what is the time read by $\mathbf{U}_B$ as $\mathbf{B}$ passes $\mathbf{O}$ (at event N) and shall compare it with the time read by the clock $\mathbf{U}_O$.

Let $\mathbf{S}''$ be an imaginary frame of reference moving with $\mathbf{B}$ and K'' be the corresponding coordinate system. According to (10) the time by $\mathbf{U}_B$ is running by the formula

$$(11)\qquad t'' = \overline{\alpha} t - \overline{\beta} x + a,$$

where a should be chosen so that the event M with its coordinates $(s, s/v_+)$ with respect to K (see Fig. 1) should have coordinates $(0, s/(v_+\gamma))$ with respect to K''. Substituting these coordinates in (11) we get $a = s(1/\gamma - \overline{\alpha} + v_+\beta)/v_+$. The event N has coordinates $(0, t_N = s/v_+ + s/v_-)$ with respect to K. Carrying out some substitutions in (11) we get that at event N the clock $\mathbf{U}_B$ reads $t''_N = s/(v_+\gamma) + s(v_-\overline{\gamma})$. The round-trip time-delation effect is given by

$$(12)\qquad r = \frac{t''_N}{t_N} = \left(\frac{1}{v_+\gamma} + \frac{1}{v_-\overline{\gamma}}\right) \Big/ \left(\frac{1}{v_+} + \frac{1}{v_-}\right) = \frac{v_+\gamma + v_-\overline{\gamma}}{\gamma\overline{\gamma}(v_+ + v_-)},$$

which gives the retardation of the clock $\mathbf{U}_B$ with respect to the clock $\mathbf{U}_O$.

We shall make use of this formula later on. It contains the undetermined parameters γ and $\overline{\gamma}$; however, the formula has the advantage of being derived before any hypotheses about the one-way velocity of light have been made. Now we are going to complete the determination of the parameters in (9) by considering some velocity of light assumptions.

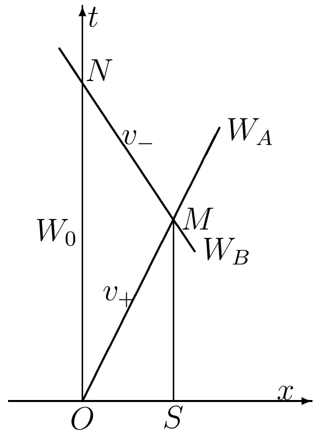

Fig. 1

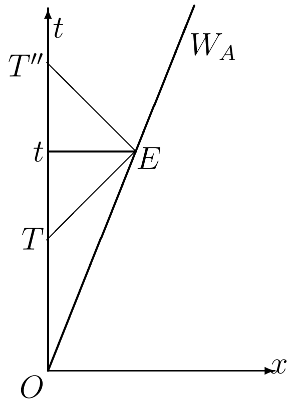

Fig. 2

4 The ε-Lorentz Transformations

Next we give Winnie's formulation of the principle of the constancy of the velocity of light. (For a discussion of the physical bases of this formulation see [1].)

Round-trip Light Principle (RTLP). The average round-trip speed of any light signal propagated (in vacuo) in a closed path is equal to the constant c in all inertial frames of reference.

According to this principle $1/\vec{c} + 1/\overleftarrow{c} = 2/c$ holds for the velocities of light in both directions with respect to K and $1/\vec{c}\,' + 1/\overleftarrow{c}' = 2/c$ holds with respect to K'. If we use Reichenbach's notation $t = T + \varepsilon(T'' - T)$, where $0 < \varepsilon < 1$ (see Fig. 2) we get $\vec{c} = c/(2\varepsilon)$, $\overleftarrow{c} = c/(2-2\varepsilon)$ and analogically, $\vec{c}\,' = c/(2\varepsilon')$, $\overleftarrow{c}' = c/(2-2\varepsilon')$. The requirement that $x = \vec{c}\, t$, $x' = \vec{c}\,' t'$ and $x = -\overleftarrow{c}\, t$, $x' = -\overleftarrow{c}' t'$ should be world-lines of one and the same signals with respect to K and K' resp., yields by means of (9) the relation $\alpha\,\vec{c}\,' + v_+\gamma = \vec{c}\,(\beta\,\vec{c}\,' + \gamma)$ and $\alpha\,\overleftarrow{c}' - v_+\gamma = \overleftarrow{c}\,(-\beta\,\overleftarrow{c}' + \gamma)$. Substituting here the expression for the velocities of light and solving for α and β we get $\alpha = (2(1-\varepsilon-\varepsilon')v_+/c + 1)\gamma$ and $\beta = c^{-2}(2c(\varepsilon - \varepsilon') + 4\varepsilon(1-\varepsilon)v_+)\gamma$. The substitution of these expressions in (8) and the assumption $\gamma > 0$ determine γ. Finally, the transformations (9) become

$$\begin{aligned} x &= [(2(1-\varepsilon-\varepsilon')v_+/c + 1)x' + v_+t']\gamma, \\ t &= [c^{-2}(2c(\varepsilon-\varepsilon') + 4\varepsilon(1-\varepsilon)v_+)x' + t']\gamma, \\ \gamma &= [(1-(2\varepsilon-1)v_+/c)^2 - (v_+/c)^2]^{-1/2}, \end{aligned} \tag{13}$$

called by Winnie ε-Lorentz transformations. As we mentioned in the introduction, we shall call the model based on (13) ε-STR.

The coefficient of (10) can be determined in the same way. It turns out that they can be obtained from the expressions for α, β and γ after one formally replaces there v_+ by $(-v_-)$. For example we have $\overline{\gamma} = [(1 + (2\varepsilon - 1)v_-/c)^2 - (v_-/c)^2]^{-1/2}$.

Our further aim is to apply the method of the coefficient k to ε-STR.

5 The k-Calculus Approach to ε-STR

Substituting $\vec{c} = c/(2\varepsilon)$ and $\overleftarrow{c} = c/(2 - 2\varepsilon)$ in (4) and (5) we get

(14)
$$\frac{v_+}{c} = \frac{1 - k_a^2}{2(\varepsilon + (1 - \varepsilon)k_a^2)} = \frac{k_r^2 - 1}{2(1 - \varepsilon + \varepsilon k_r^2)}$$

and a similar expression for v_-/c. For $\varepsilon = 1/2$ both these expressions yield equal values for v_+ and v_-. This value is exactly v (the STR velocity). Thus we get

(15)
$$\frac{v}{c} = \frac{1 - k_a^2}{1 + k_a^2} = \frac{k_r^2 - 1}{1 + k_r^2}.$$

The above equality presents a simple relation between k_a and k_r. It can be rewritten in the form

(16)
$$k_a k_r = 1.$$

As expected, relation (14) implies no restriction as regards ε; it is easy to see that (14) is algebraically equivalent to (16) (the ε's cancel). Thus between the chronometry coefficients assigned to one moving object before and after its passing the observer the simple relation (16) exists. This relation does not depend on the conventional choice of ε and hence can be regarded as synchrony-independent.

By excluding the coefficient k_a (or k_r) from (14) and (15) we get v_+ as a function of v (similarly we get v_- from (5) and (15))

(17)
$$v_+ = \frac{cv}{c + v(2\varepsilon - 1)}, \qquad v_- = \frac{cv}{c - v(2\varepsilon - 1)};$$

cf [1], p. 85.

In the same way we might present γ and $\overline{\gamma}$ as functions of v:

(18)
$$\gamma = \frac{1 + (2\varepsilon - 1)v/c}{\sqrt{1 - (v/c)^2}}, \quad \overline{\gamma} = \frac{1 - (2\varepsilon - 1)v/c}{\sqrt{1 - (v/c)^2}}.$$

For γ expressed by k_r we get

(19)
$$\gamma = \frac{1 + \varepsilon(k_r^2 - 1)}{k_r}.$$

Using (17) and (18) we get for the round-trip time dilation effect (12) in ε-STR the expression $r = \sqrt{1-(v/c)^2}$, i. e. in ε-STR the Einsteinian time dilation effect holds and r is a synchrony-independent quantity.

Next we are going to show that the two definitions of chronometric motion are equivalent in ε-STR.

Consider a k-chronometric motion according to Definition 2. We shall show that it is a k-chronometric motion according to Definition 1 (with the same chronometry coefficient). Let the point **A** recede from **O** k-chronometrically. Then we have $T'' = k^2 T$ (see Fig. 2). As far as ε-STR is concerned, the velocity of **A** is given by (14). The event E of the reflection of the light-signal from **A** has coordinates $(v_+ t, t)$ with respect to **S** and $(0, T')$ with respect to **S**$'$; **S**$'$ is the imaginary frame of reference in which **A** is at rest. From $v_+ t = \vec{c}\ (t - T)$ (see Fig. 2) and $\vec{c} = c/(2\varepsilon)$ we have $t = T/(1 - 2\varepsilon v_+/c)$. If we substitute here v_+/c by (14), we get $t = T(1 - \varepsilon + \varepsilon k^2)$. According to (13) the time interval T' measured by $\mathbf{U}_A$ between the meeting with **O** and the event **E** is $T' = t/\gamma$. When we substitute here the obtained expression for t and formula (19) for γ we get $T' = kT$, i. e. **A** moves k-chronometrically according to Definition 1.

Conversely, let **A** move k-chronometrically according to Definition 1, i. e. (1) holds. From $v_+ t = \overleftarrow{c}\ (T'' - t)$ and $\overleftarrow{c} = c(2 - 2\varepsilon)$ (using $t = \gamma T'$ and (19) we get $T'' = \gamma T'(1 + 2(1-\varepsilon)v_+/c) = kT' = k^2 T$ and that proves the equivalency of the two definitions in ε-STR (the equivalency in STR follows as a special case).

The above considerations answer the question why the experiment $\mathbf{E}^*$ (or **E**) gives nothing as regards the determination of the one-way velocity of light.

Finally, we shall use the k-calculus approach to show that the so-called apparent velocity [8] is another syncrony-independent quantity.

Let **A** be an object moving chronometrically as regards the observer **O**. Suppose **O** is able to measure the distance to **A** only by the light signals received from **A** (but is not able to dispatch signals). This distance we might call apparent distance to **A** and shall denote it by $\overline{x}$.

Remark. An example of such a situation gives an observable spherical object **A** whose radius R is known. Then the apparent distance $\overline{x}$ from **O** to **A** at a certain instant t by $\mathbf{U}_O$ can be evaluated by means of the angle subtended by **A** at $\overline{t}$. If this angle is $\alpha(\overline{t})$, then the apparent distance to **A** clearly is $\overline{x}(\overline{t}) = R\mathrm{cotg}(\alpha(\overline{t})/2)$.

The apparent velocity v can be defined as the rate of change of the apparent distance $\overline{x}$ with respect to $\overline{t}$; in our notations we have $\overline{v} = |\overline{x}(\overline{t})/\overline{t}|$ (see Fig. 3).

Assume that **A** moves in the positive direction, approaching **O**. We have then for the apparent velocity

$$\overline{v} = \left| \frac{\overline{x}(\overline{t})}{\overline{t}} \right| = \frac{|x(t)|}{|t - x(t)/\ \vec{c}\ |},$$

or, after substituting $\vec{c} = c/(2\varepsilon)$,

$$\frac{\overline{v}}{c} = \frac{c^{-1}|x(t)/t|}{|1-(2\varepsilon/c)(x(t)/t)|} = \frac{v_+/c}{1-2\varepsilon(v_+/c)}.$$

Using (17) and (14) we get

$$\frac{\overline{v}}{c} = \frac{v/c}{1-v/c} = \frac{1-k^2}{2k^2},$$

where k is the chronometry coefficient of **A**.

When **A** recedes we get similarly

$$\frac{\overline{v}}{c} = \frac{v_+/c}{1+2(1-\varepsilon)v_+/c} = \frac{v/c}{1+v/c} = \frac{k^2-1}{2k^2}.$$

In general, we can write $\overline{v}/c = |k^2-1|/(2k^2)$ and this relation shows that the apparent velocity does not depend on ε. Thus, the apparent velocity is an example of a synchrony-independent quantity, although its definition is based on a one-way experiment.

6 The Relativity Principle and the Hypothesis of Ritz

In this section we are studying a model based on the Relativity Principle (as formulated in Section 3) and the Ritz hypothesis instead of on the RTLP. The k-calculus approach turns to be very helpful in developing the consequences of these two principles. The Ritz hypothesis on the propagation of light can be stated as follows:

Hypothesis of Ritz (HR). Any light signal propagates in vacuo with constant velocity c with respect to the emitting source.

More precisely, if **A** is an inertially moving (with respect to **S**) source and if **S**$'$ is an imaginary frame of reference, in which **A** is at rest, then the signals emitted from **A** in all directions have constant velocity c with respect to **S**$'$. Needless to say, HR is not consistent with RTLP. The RTLP, as given in [1], contains the assumption that the velocity of light does not depend on the velocity of the source. In fact no difference is made in ε-STR whether a light signal is emitted (reflected) by a source at rest in **S** or by a source at rest in **S**$'$ — the velocity of the light signal is assumed to depend only on the direction of its propagation.

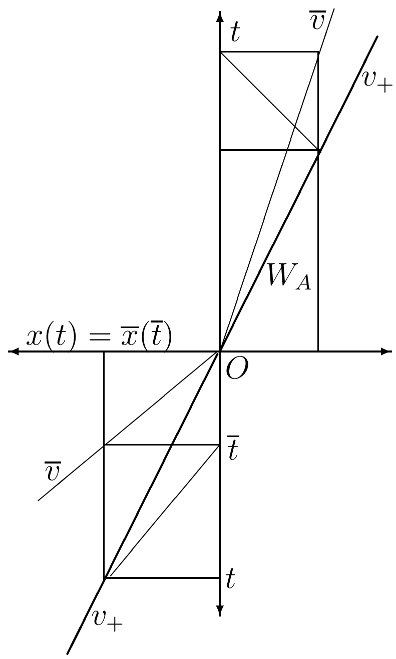

Fig. 3

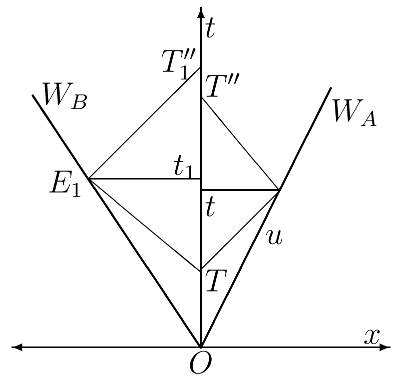

Fig. 4

As it is known, the experiments performed by now cannot refute directly the dependence of the one-way velocity of light on the velocity of the soure (in vacuo). J. G. Fox [5, 6] has shown that the so-called extinction considerations require very high vacuo for the crucial experiment (if such one exists) and that makes the task technically difficult.

On the other hand, in accordance to the Ritz hypothesis we can weaken the RTLP (retaining at the same time this part of its content which directly corresponds to the physical experiments of Michelson-Morly type) to the following:

Weak Round-trip Light Principle (WRTLP). In any inertial frame of reference **S** the average round-trip speed of any light signal propagating (in vacuo) in a closed path, the path being obtained by reflection of the signal from objects at rest in **S**, is equal to the constant c with respect to **S**.

It is easy to see that WRTLP follows directly from Ritz hypothesis. WRTLP is weaker than RTLP, because there is no restriction in RTLP for the sources (or the reflecting objects) to be at rest in the frame, in which the round-trip velocity is determined. In other words WRTLP corresponds to the results of these experiments of Michelson-Morly type, which have their reflecting mirrors at rest with respect to the frame of reference involved. Clearly RTLP assumes more than these experiments result in.

Our further aim is to develop the consequences of the Principle of Linearity, the Principle of Relativity and the hypothesis of Ritz. We shall denote by ρ-STR the obtained model because, as it will turn out, this model contains an arbitrary parameter ρ, and we shall consider this model as a form of the emission theory of Ritz (see [5]).

Let $\mathbf{S}'$ move with velocity u with respect to $\mathbf{S}$ in the positive direction (or with velocity w in negative direction; we shall assume that u and w correspond to one and the same chronometry coefficient k). Let us denote the velocities of the light signals emitted by any source at rest in $\mathbf{S}'$ by $\vec{c}_u$ (or $\vec{c}_w$) if propagating in positive direction, and by $\overleftarrow{c}_u$ (or $\overleftarrow{c}_w$) if propagating in negative direction (all these velocities are taken with respect to $\mathbf{S}$). Denote respectively by $\vec{c}\,'_u$, $\vec{c}\,'_w$, $\overleftarrow{c}\,'_u$ and $\overleftarrow{c}\,'_w$ the values of these velocities with respect to $\mathbf{S}'$. According to HR we have then $\vec{c}\,'_u=\overleftarrow{c}\,'_u= c$ and $\vec{c}\,'_w=\overleftarrow{c}\,'_w= c$ (in this section the notations u and w concern the concept of velocity as it would be in ρ-STR).

Let two points $\mathbf{A}$ and $\mathbf{B}$ move chronometrically (according to Definition 2) in opposite directions, passing each other at event $(0, 0)$ (see Fig. 4). Consider first the situation after the meeting.

Using Reichenbach's notations we have (according to Fig. 4)

$$t = T + \delta(T'' - T), \quad t_1 = T + \delta_1(T''_1 - T), \quad 0 < \delta, \delta_1 < 1. \tag{20}$$

Denote by u and w the velocities of $\mathbf{A}$ and $\mathbf{B}$ respectively. Next we shall demonstrate that if the motions of $\mathbf{A}$ and $\mathbf{B}$ have one and the same chronometry coefficient k, then $u = w$ (which was not true in ε-STR).

In fact, according to Definition 2 we have then $T'' = k^2T = T''_1$ and from (20) follows that $\delta_1(t-T) = \delta(t_1-T)$. We get from this equality and from $ut = c(t-T)$ and $wt_1 = c(t_1 - T)$ that

$$\delta_1 u(c - w) = \delta w(c - u). \tag{21}$$

The velocities $\vec{c}_w$ and $\overleftarrow{c}_u$ can be expresses as functions of δ, δ_1 and c (for example, substitute $T = t - x/c$ and $T'' = t + x/\overleftarrow{c}_u$ in (20) in order to get $\overleftarrow{c}_u$). We obtain $\overleftarrow{c}_u= (\delta(1-\delta))c$ and $\vec{c}_w= (\delta_1(1-\delta_1))c$. Substituting these values for $\overleftarrow{c}$ and $\vec{c}$ in (4) and (5) and then the obtained expressions for u and w in (21) we get $\delta_1 = \delta$. This implies $\vec{c}_w=\overleftarrow{c}_u$ and $u = w$.

Further on we shall use the notations $(1 - \delta)/\delta = \rho$ and $u = w = v_\rho$ (which we shall call ρ-STR velocity).

Consider now the situation before $\mathbf{A}$ and $\mathbf{B}$ pass each other. Replacing in (20) δ by $\overline{\delta}$ and δ_1 by $\overline{\delta}_1$ and using analogous considerations, we get again $\overline{\delta}_1 = \overline{\delta}$. We shall denote $\vec{c}_u=\overleftarrow{c}_w= c/\overline{\rho}$, where $\overline{\rho} = (1 - \overline{\delta})/\overline{\delta}$. Formula (4) becomes then

$$\frac{v_\rho}{c} = \frac{1 - k_a^2}{\rho + k_a^2} = \frac{k_r^2 - 1}{\rho + k_r^2}. \tag{22}$$

Using (15) we can express v_ρ by v

$$\frac{v_\rho}{c} = \frac{2v/c}{1 + \overline{\rho} + (\overline{\rho} - 1)v/c} = \frac{2v/c}{1 + \rho - (\rho - 1)v/c}. \tag{23}$$

The last two formulae present a relation between ρ and $\overline{\rho}$ which depend on v (or on the chronometry coefficient). Equation (23) can be rewritten more simply as

$$\rho - \overline{\rho} = (\rho + \overline{\rho} - 2)v/c. \tag{24}$$

We shall assume further on that the parameters ρ and $\overline{\rho}$ always satisfy (24). A little algebra shows that the relation $k_a k_r = 1$ holds for all values of ρ and $\overline{\rho}$ (satisfying (24)). Relation (24) can be also written as

$$\frac{\rho - 1}{\overline{\rho} - 1} = \frac{1 + v/c}{1 - v/c} = k_r^2. \tag{25}$$

The last relation shows that either both ρ and $\overline{\rho}$ are ≥ 1, or $\rho, \overline{\rho} \leq 1$. This implies that either $\delta, \overline{\delta} \geq 1/2$ or $\delta, \overline{\delta} \leq 1/2$ (assuming that $\delta, \overline{\delta} \leq 1/2$ means that in ρ-STR the velocity of any light signal does not exceed c; in that case (25) gives $\rho \geq \overline{\rho} \geq 1$).

Using the general approach outlined in Section 4, we can determine the coefficients of the transformation formulae in ρ-STR. According to (9) the velocities of the light signals with respect to K and K' are related as follows:

$$\overleftarrow{c}_u = \frac{\alpha \overleftarrow{c}'_u - v_\rho \gamma}{\gamma - \beta \overleftarrow{c}'_u}, \qquad \overrightarrow{c}_u = \frac{\alpha \overrightarrow{c}'_u + v_\rho \gamma}{\gamma + \beta \overrightarrow{c}'_u}. \tag{26}$$

Using $\overleftarrow{c}_u = c/\rho$, $\overrightarrow{c}_u = c/\overline{\rho}$ and HR (according to which $\overleftarrow{c}'_u = \overrightarrow{c}'_u = c$) and solving (26) for α and β we get

$$\alpha = \frac{2 + (\rho - \overline{\rho})v_\rho/c}{\rho + \overline{\rho}}, \quad \beta = \frac{\overline{\rho} - \rho + 2\rho\overline{\rho}v_\rho/c}{\rho + \overline{\rho}}.$$

The Principle of Relativity (8) and the standard assumption $\gamma > 0$ give for γ

$$\gamma = \left(\frac{2}{\rho + \overline{\rho}}(1 + (\rho - \overline{\rho})v_\rho/c - \rho\overline{\rho}(v_\rho/c)^2)\right)^{-\frac{1}{2}} = \left(\frac{2}{\rho + \overline{\rho}}(1 - (v_\rho/c)^2)\right)^{-\frac{1}{2}}. \tag{27}$$

Thus the transformations (9) obtain the form

$$\begin{aligned} x &= \left(\frac{2 + (\rho - \overline{\rho})v_\rho/c}{\rho + \overline{\rho}}\, x' + v_\rho t'\right)\gamma, \\ t &= \left(\frac{\overline{\rho} - \rho + 2\rho\overline{\rho}v_\rho/c}{\rho + \overline{\rho}}\, \frac{x'}{c} + t'\right)\gamma, \end{aligned} \tag{28}$$

wherein γ is given by (27).

Equations (28) become the standard Lorentz transformations for $\rho = 1$.

We can use (12) to evaluate the round-trip time dilation effect in ρ-STR. It is easily seen that $\overline{\gamma} = \gamma$; thus (12) gives

$$r = \gamma^{-1} = \left(\frac{2}{\rho + \overline{\rho}} (1 - (v_\rho / c)^2) \right)^{-1/2} .$$

This formula shows that for given velocity of the source the quantity r can be arbitrary close to the Lorentz factor $\sqrt{1 - (v/c)^2}$; the only requirement is ρ (and hence $\overline{\rho}$) to be sufficiently close to 1. Hence it can be argued that the existing experiments measuring time dilation of the life times of μ-mesons disprove ρ-STR by demonstrating a unique choice of δ. Let us note, however, that all the mathematical speculations in this section are irrelevant if the considered propagation of light signals takes place in a material medium (at rest with respect to the observer); the extinction argument (see [5, 6]) implies then that at the instances of emission the source should be regarded as at rest in relation to the medium. Therefore, we should have $\rho = \overline{\rho} = 1$, which leads immediately to STR.

In this section we have outlined an emission model based on the Relativity Principle. It turns out that this model can be regarded as a generalization of STR. The Ritz emission theory is usually regarded as diametrically opposed to Special Relativity Theory. The reason for this is that, speaking of Ritz emission theory, a theory based on non-relativistic assumptions (such as the classical concept of simultaneity) is meant, instead of a one based on the Principle of Relativity. While looking for evidence against the emission theories, it might be recommended not to ignore the fact that ρ-STR could be considered as a possible form of these theories.

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