Comptes rendus de l'Académie bulgare des Sciences
30, 9, 1977, 1239–1242

MATHÉMATIQUES
Analyse réelle

EXTENDED INTERVAL ARITHMETIC

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(Submitted by Corresponding Member Bl. Sendov on May 20, 1977)

In this note we propose an extension of interval arithmetic [2, 3] by introducing two non-standard operations for subtraction and division of intervals.

Denote by $I(\mathbb{R})$ the set of all intervals $[\alpha, \beta]$ on the real line $\mathbb{R}$. (An interval may be considered either as an element (α, β) of $\mathbb{R}^2$ with $\alpha \leq \beta$, or as a point set $\{\xi \mid \alpha \leq \xi \leq \beta\}$ in $\mathbb{R}$.) We shall denote the left endpoint of $a \in I(\mathbb{R})$ by a^- and the right endpoint of a by a^+, so that $a = [a^-, a^+]$. The length of the interval a we denote by $\mu(a) = a^+ - a^-$.

Define in $I(\mathbb{R})$ addition by means of

$$\text{(A)} \qquad a + b = [a^- + b^-,\ a^+ + b^+]$$

and scalar multiplication by

$$\text{(SM)} \qquad \alpha a = \begin{cases} [\alpha a^-,\ \alpha a^+], & \text{if } \alpha \leq 0, \\ [\alpha a^+,\ \alpha a^-], & \text{if } \alpha < 0. \end{cases}$$

The product $(-1)a$ is denoted briefly by $-a$.

The following relations hold in $I(\mathbb{R})$ with respect to the operations (A) and (SM):

1. $I(\mathbb{R})$ is a commutative semigroup with respect to (A), i. e.: (1a) $a + b = b + a$ and (1b) $(a + b) + c = a + (b + c)$;
2. $\alpha(b + c) = \alpha b + \alpha c$;
3. $(\alpha + \beta)c = \alpha c + \beta c$ for $\alpha\beta \geq 0$;
4. $\alpha(\beta c) = (\alpha\beta)c$;
5. $0a = 0$;
6. $1a = a$.

Denote the algebraic system of the set $I(\mathbb{R})$ and the operations (A) and (SM) by $I_0 = \langle I(\mathbb{R}), (A), (SM)\rangle$. I_0 is a quasilinear space in the sense of [1].

Define now a non-standard subtraction in $I(\mathbb{R})$ by means of

$$\text{(S)} \qquad a - b = [\min\{a^- - b^-, a^+ - b^+\},\ \max\{a^- - b^-, a^+ - b^+\}].$$

An equivalent definition is:

$$a - b = \begin{cases} [a^- - b^-, a^+ - b^+], & \text{if } \mu(a) \geq \mu(b), \\ [a^+ - b^+, a^- - b^-], & \text{if } \mu(a) < \mu(b). \end{cases}$$

Note that in general $a - b \neq a + (-b)$ (here, of course, $a + (-b)$ is the standard arithmetic subtraction); $a - b = a + (-b)$ iff $\mu(a)\mu(b) = 0$.

Denote the algebraic system $\langle I(\mathbb{R}), (A), (SM), (S)\rangle$ by I_1. The following relations hold in I_1 in addition to relations 1–6:

7. $(-a) - b = (-b) - a$;

8. $\alpha(a - b) = \alpha a - \alpha b$;

9. $(\alpha + \beta)c = \alpha c - (-\beta c)$ for $\alpha\beta < 0$.

Some simple corollaries are: $a - a = 0$, $a - b = -(b - a)$, $a - (-b) = b - (-a)$. Relations 3 and 9 can be combined in the general formula:

$$(\alpha + \beta)c = \begin{cases} \alpha c + \beta c, & \text{if } \alpha\beta \geq 0, \\ \alpha c - (-\beta c), & \text{if } \alpha\beta < 0. \end{cases}$$

In what follows we shall frequently make use of the function $\mu(a)$. This function satisfies the following relations:

$$\mu(a) \geq 0, \quad \mu(\alpha a) = |\alpha|\mu(a), \quad \mu(a \pm b) = |\mu(a) \pm \mu(b)|. \tag{1}$$

Note that relations 7, 8 and 9 correspond in some sense to relations 1a, 2 and 3. We give next an analogue to relation 1b (associative rule for addition and subtraction). Denote for brevity: $M_1 = (\mu(a) - \mu(c))(\mu(b) - \mu(d))$, $M_2 = (\mu(a) - \mu(b))(\mu(c) - \mu(d))$, $M_3 = (\mu(a) - \mu(d))(\mu(c) - \mu(b))$. Then we have:

10a. $$(a + b) - (c + d) = \begin{cases} (a - c) + (b - d), & \text{if } M_1 \geq 0, \\ (a - c) - (d - b), & \text{if } M_1 < 0. \end{cases}$$

10b. $$(a - b) + (c - d) = \begin{cases} (a + c) - (d + b), & \text{if } M_2 \geq 0, \\ (a - (-c)) + ((-b) - d), & \text{if } M_2 < 0,\ M_1 < 0, \\ (a - (-c)) - (b - (-d)), & \text{if } M_2 < 0,\ M_1 \geq 0. \end{cases}$$

10c. $$(a - b) - (c - d) = \begin{cases} (a + d) - (b + c), & \text{if } M_2 \geq 0, \\ (a - (-d)) - (b - (-c)), & \text{if } M_2 < 0,\ M_3 < 0, \\ (a - (-d)) - ((-b) - c), & \text{if } M_2 < 0,\ M_3 \geq 0. \end{cases}$$

As special cases we have : $(a + b) - a = b$; $(a - b) + a = a$ for $\mu(a) \geq \mu(b)$; $(a - b) - a = -b$ for $\mu(a) \geq \mu(b)$. The following corollaries hold as well:

Proposition 1 *$a + b = c$ implies $a = c - b$ and $b = c - a$.*

Proposition 2

$$c = a - b \Longleftrightarrow \begin{cases} a = c + b, & \text{if } \mu(a) \geq \mu(b), \\ a = c - (-b), & \text{if } \mu(a) < \mu(b). \end{cases}$$

Proposition 3 *The equation $a + x = b$ has a solution if $\mu(a) \leq \mu(b)$. In this case the unique solution is $x = b - a$.*

In particular the equality $a + x = 0$ has a solution if and only if $\mu(a) = 0$; the solution is then $x = 0 - a = -a$.

We shall introduce a norm in I_1 by

$$||a|| = \max\{|a^-|, |a^+|\}.$$

Then we have obviously $||a|| = r(a, 0)$ and $||a - b|| = r(a, b)$, where $r(a, b)$ is the Hausdorff distance between a and b:

$$r(a, b) = \max\{|a^- - b^-|, |a^+ - b^+|\}.$$

Remark: It may be of interest to consider abstract quasilinear spaces with three operations which satisfy by definition relations 1–9 (or relations 1–10; in this case a function μ should be also defined by means of (1)). In such quasilinear spaces one can introduce a norm; and then study normed quasilinear spaces.

The (standard) operation for multiplication in $I(\mathbb{R})$ is

$$\text{(M)} \qquad ab = [\min\{a^-b^-, a^-b^+, a^+b^-, a^+b^+\},\ \max\{a^-b^-, a^-b^+, a^+b^-, a^+b^+\}].$$

The scalar multiplication (SM) is, of course, a special case of (M).

In order to formulate an equivalent definition of (M), which is more useful for practical computations, we introduce some further notations. Given $a \in I(\mathbb{R})$, denote by $\tilde{a}$ the endpoint of a which has maximum absolute value (for example, if $a = [-5, 3]$, then $\tilde{a} = -5$). Given $a, b \in I(\mathbb{R})$, we denote (as in [3]) the interval $[\min(a^-, b^-), \max(a^+, b^+)]$ by $a \bigvee b$. We shall say that a and b are of equal signs if either $a^- > 0$, $b^- > 0$ or $a^+ < 0$, $b^+ < 0$; a and b are of opposite signs if either $a^- > 0$, $b^+ < 0$ or $a^+ < 0$, $b^- > 0$. The following definition is equivalent to (M) in the case $b \not\ni 0$:

$$\text{(M}'\text{)} \qquad ab = \begin{cases} a^-b^- \bigvee a^+b^+, & \text{if } a, b \text{ are of equal signs,} \\ a^-b^+ \bigvee a^+b^-, & \text{if } a, b \text{ are of opposite signs,} \\ (\tilde{b})a, & \text{if } a \ni 0. \end{cases}$$

Let us introduce now a non-standard division in $I(\mathbb{R})$. For $a, b \in I(R)$, $0 \notin b$, we define:

$$\text{(D)} \qquad a/b = \begin{cases} a^-/b^- \bigvee a^+/b^+, & \text{if } a, b \text{ are of equal signs,} \\ a^-/b^+ \bigvee a^+/b^-, & \text{if } a, b \text{ are of opposite signs,} \\ (1/(\tilde{b}))a, & \text{if } a \ni 0. \end{cases}$$

The reader may note the similarity between (D) and (M′).

It can be easily shown that $a(1/b)$ means standard arithmetic division: $a(1/b) = \{\xi/\eta \mid \xi \in a, \eta \in b\}$. In general $a/b \neq a(1/b)$; we have $a/b = a(1/b)$ if and only if $\mu(a)\mu(b) = 0$.

Denote by I_2 the algebraic system $\langle I(\mathbb{R}), (A), (S), (M), (D)\rangle$. Here are some properties of I_2.

Proposition 4 *Let $a, b, c \in I(\mathbb{R})$, $ab > 0$ and $0 \notin abc$. Then we have:*

$$(a \pm b)c = ac \pm bc, \quad (a \pm b)/c = a/c \pm b/c.$$

Proposition 5 *If $0 \notin b$, then $(ab)/b = a$.*

In particular we have $a/a = 1$ $(0 \notin a)$.

Proposition 6 *$ab = c$ implies $a = c/b$ and $b = c/a$.*

For $a \in I(\mathbb{R})$, $0 \notin a$, define the function ν by

$$\nu(a) = \begin{cases} a^+/a^-, & \text{if } a^- > 0, \\ a^-/a^+, & \text{if } a^+ < 0. \end{cases}$$

The function ν has the following properties:

$$\nu(a) \geq 1,$$
$$\nu(ab) = \nu(a)\nu(b),$$
$$\nu(a/b) = |\nu(a)/\nu(b)|^*, \quad |\alpha|^* = \begin{cases} \alpha, & \text{if } \alpha \geq 1, \\ 1/\alpha, & \text{if } 0 < \alpha < 1. \end{cases}$$

Proposition 7

$$c = a/b \Longleftrightarrow \begin{cases} a = cb, & \text{if } \nu(a) \geq \nu(b), \\ a = c/(1/b), & \text{if } \nu(a) < \nu(b). \end{cases}$$

Proposition 8 *Let $a, b \in I(\mathbb{R})$, $0 \notin a$. If $0 \in b$, the equation (E) $ax = b$ has a unique solution $x = b/a$. For $0 \notin b$ (E) has a unique solution if and only if $\nu(a) \leq \nu(b)$. In that case the solution is again given by $x = b/a$.*

Finally we shall give a possible application of the extended interval arithmetic.

Given a real rational function $\varphi(\xi_1, \xi_2, \ldots, \xi_n)$, it is asked to find the range f of values of φ, when ξ_i varies in given intervals $x_i \in I(\mathbb{R})$, $i = 1, \ldots, n$.

In some simple cases (when each ξ_i appears only once and to the first power in φ) we can solve this problem by means of the standard interval arithmetic. For example, given $x_i \in I(\mathbb{R})$, $i = 1, \ldots, 4$, we can write:

$$\left\{ \varphi = \frac{2\xi_1 - 3\xi_2}{\xi_3 + \xi_4} \mid \xi_i \in x_i \right\} = (2x_1 + (-3x_2))(1/(x_3 + x_4)),$$

replacing the variables ξ_1 by x_1, ξ_2 by x_2, etc. and the operations in the expression for φ by standard arithmetic operations between the corresponding intervals. The interval expression thus obtained can be easily evaluated by means of interval arithmetic.

More generally, given a real function $\varphi = \varphi(\xi_1, \ldots, \xi_n)$ and $x_1, \ldots, x_n \in I(\mathbb{R})$, we want to be able to write an interval expression for the range of φ as ξ_i vary in x_i. We hope that our arithmetic extends the possibilities for treating such problems. In particular, we hope that the theory of matrix computations with intervals (see [2], Ch. 5) can be refined when extended interval arithmetic is used.

As an example consider the rational expression $\varphi(\xi) = (\alpha\xi + \beta)/(\gamma\xi + \delta)$, wherein the variable $\xi \in x \in I(\mathbb{R})$ occures twice. Assume that $\alpha\delta - \beta\gamma \neq 0$ and $0 \notin \gamma x + \delta$. Assume for simplicity $0 \notin \alpha x + \beta$ as well, so that for $\xi \in x$ we have $\mathrm{sign}\varphi(\xi) = \mathrm{const} = \sigma \in \{-1, 1\}$. It is easily seen then, that

$$\{\varphi(\xi) \mid \xi \in x\} \subset (\alpha x + \beta)(1/(\gamma x + \delta)).$$

The sign "$\subset$" cannot be replaced by "$=$" in general. However, using extended inteval arithmetic we are able to obtain an equality relation, namely we can state:

$$\{\varphi(\xi) \mid \xi \in x\} = \begin{cases} (\alpha x + \beta)(1/(\gamma x + \delta)), & \text{if } \mathrm{sign}(\alpha\gamma) = \sigma, \\ (\alpha x + \beta)/(\gamma x + \delta), & \text{if } \mathrm{sign}(\alpha\gamma) \neq \sigma. \end{cases}$$

Note that in the case $\mathrm{sign}(\alpha\gamma) = \sigma$ standard arithmetic division is used, whereas by $\mathrm{sign}(\alpha\gamma) \neq \sigma$ the division is non-standard.

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