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A Method for Solving Algebraic Equations

In this paper we consider a numerical method for solving algebraic equations with real distinct roots, which has been first announced in [4]. The method makes use of advanced computer arithmetic and has been recently realised in the frames of two program systems, which provide such arithmetic: PASCAL-SC and HIFICOMP.

Introduction

Consider an algebraic polynomial f of degree n, possessing n distinct zeroes. This means that f possesses a representation of the form

$$f(z) = (z - \zeta_1)(z - \zeta_2) \dots (z - \zeta_n) = \prod_{i=1}^{n} (z - \zeta_i) , \tag{1}$$

where $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)$ is a complex vector such that $\zeta_i \neq \zeta_j$ for all $i, j \in \{1, \dots, n\}$ with $i \neq j$. We assume that ζ is unknown and we want to determine the vector ζ assuming additionally that the value $f(z)$ can be effectively computed (e.g. with maximum accuracy in the sence of [5]) for any z from some domain containing the vector ζ.

Remark: In practical problems the algebraic polynomial can be given in various ways. For instance, it can be given in the form $f(z) = \sum_{i=0}^{n} a_i z^i$, $a_n = 1$, or it can be given as the characteristic equation of a given matrix, etc.

The idea of the simultaneous determination of all roots of an algebraic polynomial goes back to K. Weierstrass. In more recent time this idea has been exploited and developed by K. Dočev. He considers an iteration procedure (see e.g. [2]), which can be written in ALGOL-like form as

$$z_i := z_i - f(z_i)/\{(z_i - z_1)(z_i - z_2) \dots (z_i - z_{i-1})(z_i - z_{i+1}) \dots (z_i - z_n)\} , \tag{2}$$

$i = 1, \dots, n$. It is assumed thereby that an appropriate initial approximation $z^{(0)} = (z_1^{(0)}, \dots, z_n^{(0)})$ is given.

In [4], p. 129, we prove a theorem for the quadratic convergence of the iteration procedure (2). This theorem is a refinement of a similar theorem formulated in the textbook [7], p. 206. If we denote $d = \min_{i \neq j} |\zeta_i - \zeta_j|$ and if $z^{(k)} = (z_1^{(k)}, \dots, z_n^{(k)})$ denotes the result obtained on the k-th step of the iteration procedure (2), then the theorem can be formulated as follows.

Theorem 1: *Let q and c be two reals such that $0 < q < 1$, $0 < c \leq d/(1 + \alpha n)$, $\alpha = 1.7632 \dots$ being defined by $\alpha = \exp(-\alpha)$ and the initial approximation $z^{(0)} = (z_1^{(0)}, \dots, z_n^{(0)})$ be such that $|z_i^{(0)} - \zeta_i| \leq cq$. Then the approximations $z_i = z_i^{(k)}$ produced by (2) on the k-th iteration step satisfy the inequalities*

$$|z_i^{(k)} - \zeta_i| \leq cq^{2^k} , \quad i = 1, \dots, n , \quad \textit{for all} \quad k = 1, 2, \dots .$$

We have further developed the above ideas by constructing an interval numerical method [4], producing bounds for the unknown roots under the additional assumption that the roots are real. We shall further denote the roots of (1) by $\xi = (\xi_1, \dots, \xi_n)$ (instead of ζ), and the argument of f by $x = (x_1, \dots, x_n)$ (instead of z) in order to stress on the fact that only real valued roots and arguments (and, consequently, polynomial values) will be involved. In what follows we describe the interval numerical method suitably furnished by the necessary computer-arithmetic operations and report on two recently obtained program realizations of this method and on some numerical experiments. The algorithms, resp. the computer programs, deliver guaranteed bounds with maximum accuracy.

The algorithm

Assume that i) $\underline{x}^{(0)} = (\underline{x}_1^{(0)}, \dots, \underline{x}_n^{(0)})$ are given left approximations of the roots ξ_i; ii) $\bar{x}^{(0)} = (\bar{x}_1^{(0)}, \dots, \bar{x}_n^{(0)})$ are right approximations of the roots ξ_i, and iii) the intervals $X_i^{(0)} = [\underline{x}_i^{(0)}, \bar{x}_i^{(0)}]$ do not overlap, that is

$$X_i^{(0)} \cap X_j^{(0)} = \emptyset \quad \text{for all} \quad i \neq j , \quad \text{and} \quad \xi_i \in X_i^{(0)} , \quad i = 1 , \dots , n.$$

Consider the following iteration procedure (written in ALGOL-like form):

$$\left.\begin{aligned} \underline{x}_i &:= \underline{x}_i - f(\underline{x}_i)/\{(\underline{x}_i - \underline{x}_1)(\underline{x}_i - \underline{x}_2) \dots (\underline{x}_i - \underline{x}_{i-1})(\underline{x}_i - \bar{x}_{i+1}) \dots (\underline{x}_i - \bar{x}_n)\} , \\ \bar{x}_i &:= \bar{x}_i - f(\bar{x}_i)/\{(\bar{x}_i - \underline{x}_1)(\bar{x}_i - \underline{x}_2) \dots (\bar{x}_i - \underline{x}_{i-1})(\bar{x}_i - \bar{x}_{i+1}) \dots (\bar{x}_i - \bar{x}_n)\} , \end{aligned}\right\} \tag{3}$$

$i = 1, 2, \dots, n$. Denote $d = \min_{i \neq j} |\xi_i - \xi_j|$. The following theorem holds true [4].

Theorem 2: *If the real numbers q, c, $\{\underline{x}_i^{(0)}\}_{i=1}^n$, $\{\bar{x}_i^{(0)}\}_{i=1}^n$ are such that $0 < q < 1$, $0 < c < d/n$, $0 \leq \xi_i - \underline{x}_i^{(0)} \leq cq$, $0 \leq \bar{x}_i^{(0)} - \xi_i \leq cq$, then the one-sided approximations $\underline{x}_i^{(k)}$, $\bar{x}_i^{(k)}$, produced by (3) on the k-th iteration, satisfy the inequalities*

$$0 \leq \xi_i - \underline{x}_i^{(k)} \leq cq^{2k} , \quad 0 \leq \bar{x}_i^{(k)} - \xi_i \leq cq^{2k} , \quad i = 1, 2, \dots, n , \quad \textit{for all} \quad k = 1, 2, \dots .$$

We can now furnish the formula (3) with advanced computer-arithmetic operations so that the computed intervals $[\underline{x}_i^{(k)}, \overline{x}_i^{(k)}]$ contain the roots ξ_i with guarantee. We have

$$\left.\begin{aligned} &\underline{x}_i := \underline{x}_i {}^{-}\nabla(\square f(\underline{x}_i)'\nabla \square_0\, g(\underline{x}_i))\,, \qquad \overline{x}_i := \overline{x}_i \Delta^{-}(\square f(\overline{x}_i)\, \nabla' \square_0 g(\overline{x}_i))\,, \\ &g(x) = (x - \underline{x}_1)\,(x - \underline{x}_2) \ldots (x - \underline{x}_{i-1})\,(x - \overline{x}_{i+1}) \ldots (x - \overline{x}_n)\,. \end{aligned}\right\} \tag{4}$$

In (4) ${}^{-}\nabla$, ∇^{-} mean subtraction to the left, resp. to the right; $'\nabla$, Δ' mean division to the left, resp. to the right, $\square$ means rounding towards zero and $\square_0$ means rounding away from zero (for more details on the advanced computer-arithmetic operations see [5]). The products $g(\underline{x}_i)$, $g(\overline{x}_i)$ are to be computed with maximum accuracy. To this end we have used a special program, although a general algorithm for highly accurate computation of rational expressions can be also applied [5].

Numerical experiments

We have realised an algorithm based on formula (4) using thereby as terminating criteria the effect of finite convergence as it is usually used in interval algorithms. The corresponding program has been included as a subroutine in the program package HIFICOMP, which involves advanced computer-arithmetic operations in the form of subroutines [3]. We have also written a corresponding program in the PASCAL-SC system for IBM PC-XT [6]. Below a computer experiment with the latter program is presented.

Numerical example: The following example is taken from [1]. The eigenvalues of the matrix

$$A = \begin{pmatrix} 12 & 1 & 0 & 0 & 0 \\ 1 & 9 & 1 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

have to be computed. As initial intervals we have used the intervals for the roots of the characteristic equation $f(x) \equiv \det(A - xE) = 0$ produced by the Theorem of Gerschgorin. We give below the final result obtained by the PASCAL-SC program on the 7-th iteration step:

$$\begin{aligned} X_1^{(0)} &= [-1, 1]\,, & X_1^{(7)} &= -0.3168759526_3^1\,, \\ X_2^{(0)} &= [1, 5]\,, & X_2^{(7)} &= 2.9838636968_3^5\,, \\ X_3^{(0)} &= [4, 8]\,, & X_3^{(7)} &= 6.0000000000_{-1}^1\,, \\ X_4^{(0)} &= [7, 11]\,, & X_4^{(7)} &= 9.0161363031_5^7\,, \\ X_5^{(0)} &= [11, 13]\,, & X_5^{(7)} &= 12.2316875952_6^7\,. \end{aligned}$$

We see that in this example the final results are obtained with an accuracy of 2 ulp's (ulp means „unit" in the last position). This is because in this example the values of f have been computed with an accuracy of 2 ulp's. Further experiments showed that the accuracy of the final results is equal to the accuracy of the computed value of f.

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