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STOCHASTIC ARITHMETIC, THEORY AND EXPERIMENTS*

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Abstract. Stochastic arithmetic has been developed as a model for exact computing with imprecise data. Stochastic arithmetic provides confidence intervals for the numerical results and can be implemented in any existing numerical software by redefining types of the variables and overloading the operators on them. Here some properties of stochastic arithmetic are further investigated and applied to the computation of inner products and the solution to linear systems. Several numerical experiments are performed showing the efficiency of the proposed approach.

1. Introduction. Stochastic arithmetic has been developed as a model for computing with imprecise data when the data belong to some known Gaussian distribution $N(\mu,\ \sigma)$. It is an old idea [8] and has been first formalized by J. Vignes and J. M. Chesneaux as a theoretical approach of the Cestac method [5], [9], [10]. In the scope of stochastic arithmetic, imprecise data are interpreted as

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stochastic numbers and the computation on them is called stochastic arithmetic. Thus, stochastic arithmetic models operations on Gaussian distributions and provides confidence intervals for the results of numerical computation in the same way that interval arithmetic provides bounds for these results. It must be also remarked that the operations of stochastic arithmetic resemble the operations for intervals whenever the intervals are presented in the midpoint-radius form. Some algebraic properties of stochastic numbers have been further investigated. In particular we give a formal definition of narrow stochastic numbers. It is shown that the standard deviation on the result of an n-dimensional inner product of two vectors with imprecise data is proportional to $\sqrt{n}$. An experimental software has been developed which implements stochastic arithmetic. This software allows to run very easily standard code written in Fortran or in C++ with stochastic arithmetic. Numerical experiments for the computation of inner products and the solution of linear systems with imprecise data are reported. In each case a confidence interval for the solution (or for the components of the solution) is provided. The obtained results confirm the theory and show that a straightforward use of stochastic arithmetic may easily lead to instructive features of a numerical problem such as the detection of instabilities.

2. Stochastic numbers and stochastic arithmetic.

As said before, stochastic arithmetic is a model for exact computation on imprecise data. Let us consider imprecise data as a Gaussian distribution with known mean value m and known standard deviation σ.

Definition 1. *The set of stochastic numbers denoted $\mathbb{S}$ is the set of Gaussian random variables.*

Thus an element $X \in \mathbb{S}$ is a pair $X = (m, \sigma)$, m being the mean value of X and $\sigma \geq 0$ its standard deviation. The main property of a Gaussian distribution and hence of a stochastic number is: For $X = (m, \sigma)$ there exists λ_η such that

$$\mathcal{P}\left(X \in [m - \lambda_\eta \sigma, m + \lambda_\eta \sigma]\right) = 1 - \eta, \tag{1}$$

where $\mathcal{P}$ denotes a probability, and $[m - \lambda_\eta \sigma, m + \lambda_\eta \sigma]$ is the confidence interval of m with a probability $(1 - \eta)$. It is well known that for $\eta = 0.05$, that is $\mathcal{P} = 1 - \eta = 0.95$, we have $\lambda_\eta = 1.96$. Consequently the number of significant decimal digits on m is the integer part of:

$$C_{\eta,X} = \log_{10}\left(\frac{|m|}{\lambda_\eta \sigma}\right), \tag{2}$$

providing $|m|/(\lambda_\eta \sigma) \geq 10$, otherwise we assume $C_{\eta,X} = 0$.

This simple but important property is used in the Cestac method and in the Cadna software to compute the number of significant digits of numerical results. The ratio $|m|/(\lambda_\eta \sigma)$ will be called relative error or relative accuracy of the stochastic number X. This characteristic is analogous to the relative error of an approximate number.

The arithmetic operations on stochastic numbers are defined as the operations on independent Gaussian distributions. They are denoted ${}_{\mathrm{s}}+$, ${}_{\mathrm{s}}-$, ${}_{\mathrm{s}}*$, ${}_{\mathrm{s}}/$ and are:

$$X_1 \ {}_{\mathrm{s}}+ \ X_2 \stackrel{def}{=} \left(m_1 + m_2, \ \sqrt{\sigma_1^2 \ + \sigma_2^2} \right),$$

$$X_1 \ {}_{\mathrm{s}}- \ X_2 \stackrel{def}{=} \left(m_1 - m_2, \ \sqrt{{\sigma_1}^2 \ + {\sigma_2}^2} \right),$$

$$X_1 \ {}_{\mathrm{s}}* \ X_2 \stackrel{def}{=} \left(m_1 m_2, \ \sqrt{{m_2}^2{\sigma_1}^2 + {m_1}^2{\sigma_2}^2 \ + \sigma_1^2 \ \sigma_2^2} \right),$$

$$X_1 \ {}_{\mathrm{s}}/ \ X_2 \stackrel{def}{=} \left(m_1/m_2, \sqrt{\left(\frac{\sigma_1}{m_2}\right)^2 + \left(\frac{m_1\sigma_2}{m_2^2}\right)^2 + \left(\frac{\sigma_1\sigma_2}{m_2^2}\right)^2} \right), \ m_2 \neq 0.$$

Algebraic properties. We next give some particular cases from the stochastic arithmetic formulae; below $X = (m, \sigma)$. We have $X_{\mathrm{s}}{+}X = (2m, \sqrt{2}\sigma)$ and $X_{\mathrm{s}}{*}X = (m^2, \sqrt{2m^2\sigma^2 + \sigma^4}) = (m^2, \sigma\sqrt{2m^2 + \sigma^2})$.

Note that the latter formula is computed as if the two variables X were independent, that is computing $X_{\mathrm{s}}{*}Y$ and then replacing Y by X exactly as in the case of intervals. It is well known that the correct formula for the computation of the mean value and standard deviation of the square of a centred random variable is: $X^2 = (m^2 + \sigma^2, \sqrt{4m^2\sigma^2 + 2\sigma^4}) = (m^2 + \sigma^2, \sigma\sqrt{4m^2 + 2\sigma^2})$.

A real number $c \in \mathbb{R}$ is identified as the degenerate stochastic number $(c, 0)$. So, in particular, for $c \in \mathbb{R}$, we have $c_{\mathrm{s}}{*}(m, \sigma) = (c, 0)_{\mathrm{s}}{*}(m, \sigma) = (cm, |c|\sigma)$. Thus multiplication of a stochastic number $X = (m, \sigma)$ by a scalar $\gamma \in \mathbb{R}$ is given by: $\gamma \ {}_{\mathrm{s}}* \ X = (\gamma, 0) \ {}_{\mathrm{s}}* \ (m, \sigma) = (\gamma m, |\gamma|\sigma)$.

Note that $X_{\mathrm{s}}{+}X = (2m, \sqrt{2}\sigma)$ is different from $2_{\mathrm{s}}{*}X = (2m, 2\sigma)$.

We have $1/X = (1, 0)_{\mathrm{s}}/(m, \sigma) = (1/m^2)_{\mathrm{s}}{*}X$ showing that inversion is reduced to multiplication by scalar. Similarly, division of two stochastic numbers is reduced to multiplication: $X_{1\mathrm{s}}/X_2 = X_{1\mathrm{s}}{*}(1_{\mathrm{s}}/X_2)$.

Note also that $0_s - X = (0,0)_s - (m,\sigma) = (-m,\sigma)$, which equals $-1_s * X$.

Many of the properties of stochastic arithmetic and stochastic numbers have been studied from the point of view of abstract algebraic structures, especially with respect to the operations addition and multiplication by scalars [1], [3], [6], [7], in particular we have:

• The set of stochastic numbers is a monoid with respect to addition. This monoid can be extended to a group structure (admitting thus negative values for σ).

• Multiplication by a scalar induces a structure of S-space which is close to a vector space and computations in S-spaces are reduced to computations in vector spaces.

As a consequence of the S-space structure it has been shown in [4] that the confidence interval of the result of an inner product of two n-dimensional vectors, one with exact data and the other with imprecise data, increases proportionally to $\sqrt{n}$. Let us show now that this is also true for the inner product of two vectors of stochastic numbers.

Let P be an inner product $P = \sum_{i=1}^{i=n} X_i \ {}_s{*} \ Y_i$ with $X_i = (m_i, \sigma_i)$ and $Y_i = (r_i, \tau_i)$. From the definition of stochastic addition and multiplication, $P = (p, \theta)$ with:

$$p = \sum_{i=1}^{i=n} m_i \ r_i, \quad \theta = \sqrt{\sum_{i=1}^{i=n} (m_i^2 \tau_i^2 + r_i^2 \sigma_i^2 + \sigma_i^2 \tau_i^2)}.$$

In numerical computations stochastic numbers are used to model uncertainties in input data. In this case commonly we work with "very" narrow stochastic numbers, i.e. such that $|\sigma_i/m_i| \ll 1, |\tau_i/r_i| \ll 1$. Then we have

$$\theta^2 = \sum_{i=1}^{i=n} m_i^2 r_i^2 \left(\frac{\sigma_i^2}{m_i^2} + \frac{\tau_i^2}{r_i^2} + \frac{\sigma_i^2 \tau_i^2}{m_i^2 r_i^2} \right) \approx \sum_{i=1}^{i=n} m_i^2 r_i^2 \left(\frac{\sigma_i^2}{m_i^2} + \frac{\tau_i^2}{r_i^2} \right). \tag{3}$$

Suppose now that all relative errors are identical: $|\sigma_i/m_i| = |\tau_i/r_i| = \delta$, then $\theta \approx \sqrt{2}\ \delta \sqrt{\sum_{i=1}^{n} m_i^2 \tau_i^2}$ and in the case when all terms $m_i r_i$ can be replaced by their mean value noted q then:

$$\theta \approx \sqrt{2}\ \delta\ q\ \sqrt{n}. \tag{4}$$

Equation (4) shows clearly that when the hypotheses on narrowness of stochastic numbers prove valid, see formula (5) below, and the terms $m_i r_i$ are of the same order of magnitude, which is often the case, the standard deviation of the result of an inner product with constant relative uncertainty on the data increases as $\sqrt{n}$. It must be noted that with the same hypotheses the result increases proportionally to n. Then the relative error on the result decreases when the dimension increases: $\theta/p \approx \sqrt{2}\,\delta/\sqrt{n}$. This differs from the behavior of the relative error of approximate numbers and may look strange but is confirmed by the experiments, see Table 1.

In order to give a more formal definition of "narrow stochastic number" and to give a criterion for narrowness we recall the notion of stochastic zero [9]:

Definition 2. *A stochastic number $X = (m, \sigma) \in \mathbb{S}$ is called stochastic zero, denoted $X = \underline{0}$ if $|m|/(\lambda_\eta \sigma) \le 1$.*

According to (2) a stochastic zero has no significant digits. The stochastic equality of two stochastic numbers is defined by $X_1 \ {}_{\rm s}{=} X_2 \Leftrightarrow X_1 \ {}_{\rm s}{-}\ X_2 = \underline{0}$.

Intuitively a "narrow stochastic number" should possess $k \ge 1$ significant decimal digits, where depending on the nature of the computations k can be some integer greater than or equal to one. This means that we should have

$$|m|/(\lambda_\eta \sigma) \ge 10^k. \tag{5}$$

This formula (5) provides an easy criterion for the narrowness of a stochastic number. As a correct computation on numbers requires at least one significant digit on each operand, k is often chosen $k = 1$.

Normally the number of significant digits in the results diminishes in the course of computations. We demonstrate this with the computation of $X {}_{\rm s}{*} X$. According to (2) the number of significant digits in $X = (m, \sigma)$ is the integer part of the decimal logarithm of $L_X = |m|/(\lambda_\eta \sigma)$. For $Y = X {}_{\rm s}{*} X = (m^2, \sigma\sqrt{2m^2 + \sigma^2})$ we have $L_Y = L_X/\sqrt{2 + (\sigma/m)^2} < L_X$.

In practice, when computing with narrow stochastic numbers, we can use the following approximate formulae for multiplication and division:

$$X_1 \ {}_{\rm s}{*}\ X_2 \approx \left(m_1 m_2,\ \sqrt{{m_2}^2{\sigma_1}^2 + {m_1}^2{\sigma_2}^2}\right),$$

$$X_1 \ {}_{\rm s}{/}\ X_2 \approx \left(m_1/m_2, \sqrt{\left(\frac{\sigma_1}{m_2}\right)^2 + \left(\frac{m_1\sigma_2}{m_2^2}\right)^2}\right),\ m_2 \ne 0.$$

Let us consider now a practical approach to numerical computations with stochastic numbers, called discrete stochastic numbers.

3. Discrete stochastic numbers and related software. As seen above a stochastic number is a Gaussian function with known mean-value m and standard deviation σ. Such a function can be approximated by N random Gaussian samples $x_1, x_2, \ldots, x_N$ with mean value $\overline{x}$ and empirical standard deviation s. In our scope, $(x_1, x_2, \ldots, x_N)$ is called "discrete stochastic number". The Cestac method and the Cadna software which have been developed to estimate the number of significant digits on the result of a numerical computation are based on this approach. Thus, each operation with discrete stochastic operands can be done by generating N Gaussian samples with known mean value and standard deviation for each operand and computing the empirical mean value and standard deviation of the N^2 samples representing the result.

In the Cestac method $N = 3$ and the result of each operation is also computed with 3 samples, as the method is only concerned with the number of significant digits on m. The method also chooses at random with equal probability the rounding up or down of all the numbers involved in the computation and thus takes into account both data errors and round-off errors.

Hence there are two possibilities when developing a software using stochastic numbers and stochastic arithmetic: either use the theoretical definitions and formulae of Section 2 or the discrete stochastic numbers and the approximations of the mean value and standard deviation that they provide. It is clear that the first is much faster but in our tests both approaches have been experimented with and it has been observed that they give very close results.

The experimental software for computing with stochastic numbers has been written in Fortran 90 and in C++. In both languages a new type `double_st` has been defined.

All arithmetic operations $+$, $-$, $*$, $/$ and comparison operators $\leq$, $<$, $=$, $\geq$, $>$ have been overloaded by operators ${}_{\mathrm{s}}+$, ${}_{\mathrm{s}}-$, ${}_{\mathrm{s}}*$, ${}_{\mathrm{s}}/$, ${}_{\mathrm{s}}\leq$, ${}_{\mathrm{s}}<$, ${}_{\mathrm{s}}=$, ${}_{\mathrm{s}}\geq$, ${}_{\mathrm{s}}>$. The assignment operator and input/output have been redefined. Some (but not yet all) of the standard mathematical functions have been redefined. The software offers two ways for computing with stochastic numbers:

- The above double precision software emulating the computation with (exact) stochastic numbers.
- A Monte Carlo-type synchronous computation with discrete stochastic numbers generating N random samples in the range of each data and computing the mean-value and standard deviation of all intermediate result and of the final result.

4. Numerical experiments. Three numerical experiments are described below.

• Experiment 1: Computation of sums of n uncertain positive numbers in a known range for different values of n. In this experiment the stochastic numbers are approximated by discrete stochastic numbers of $N = 30$ samples generated in the range of uncertainty with a Gaussian distribution. The mean value and standard deviation on the result are computed from the samples. The relative uncertainty on the data is = 0.001 and the numbers are generated in $[0, 100]$. The result, the relative error, the number of significant digits and the ratio "standard deviation/$\sqrt{n}$" are reported in Table 1. It can be seen that the standard deviation on the sum increases as $\sqrt{n}$. This conforms to theory [2].

Table 1. Sum of n imprecise numbers, relative uncertainty 0.001

n	Result	relat. error	ndig.	Std. dev/$\sqrt{\text{n}}$
10	$0.56501328D+03$	$0.186E-04$	4.73	$0.455E-01$
100	$0.49115716D+04$	$0.184E-04$	4.74	$0.500E-01$
1000	$0.51429866D+05$	$0.687E-05$	5.16	$0.472E-01$
10000	$0.49883813D+06$	$0.147E-05$	5.83	$0.541E-01$
100000	$0.49863373D+07$	$0.437E-06$	6.36	$0.532E-01$

• Experiment 2: Computation on inner products. The same inner products have been computed with the stochastic arithmetic software, and with discrete stochastic numbers. The data have been generated in $[-100, 100]$ with relative Gaussian uncertainties 0.01. The results and the standard deviation on it computed by the three different ways above (special software with stochastic operations, theoretical formula 3 and discrete stochastic numbers) are reported in Table 2. Here again it can be seen that the ratio standard deviation/$\sqrt{n}$ is almost constant.

• Experiment 3: Gaussian elimination with stochastic arithmetic and with the Cestac method. The system to be solved and the obtained solutions are in Table 3. The computed standard deviations provided by the theoretical stochastic operations lead to the number of significant digits provided by the Cestac method.

Table 2. Computation of inner products, relative uncertainty 0.01

n	Result	Computed Standard deviation			
		St. Ar.	Theor. form.	Discr. St. Ar.	$\theta\sqrt{n}$
10	$-7.67D+03$	$1.22E+02$	$1.22E+02$	$1.65E+02$	38.6
100	$+2.52D+04$	$4.69E+02$	$4.69E+02$	$5.69E+02$	46.9
1000	$-4.87D+04$	$1.52E+03$	$1.52E+03$	$1.87E+03$	48.1
10000	$+1.27D+05$	$4.64E+03$	$4.65E+03$	$6.71E+03$	46.5
100000	$-4.80D+05$	$1.47E+04$	$1.48E+04$	$5.08E+04$	46.8

Table 3. Experiment 3, linear system and solutions

$$\begin{pmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 23 \\ 32 \\ 33 \\ 31 \end{pmatrix}$$

Stochastic Arithm.		Cestac method
value	Std. dev.	Value with exact dig.
1.0000000000001883	$2.01E-04$	$0.9999E+000$
0.9999999999998863	$1.39E-04$	$0.1000E+001$
0.9999999999999518	$2.21E-04$	$0.1000E+001$
1.0000000000000289	$1.30E-05$	$0.99999E+000$

5. Conclusion. Stochastic arithmetic has interesting algebraic structures and leads to results with confidence intervals. It can be easily and effectively applied to numerical problems with imprecise data. Two different approaches are possible: A software with stochastic operations overloading standard operations, in this case round-off errors are ignored, or Discrete stochastic numbers as done in the Cestac method and Cadna software. In this last case round-off errors are taken into account. Experiments in the linear case show that in many problems both approaches give similar results provided that the problems are stable and that uncertainties on data are much greater than the errors due to rounding. On the other hand, it is clear that the theoretical formulae of Section 2 lead to a much smaller computing time.

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