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The Contribution of T. Sunaga to Interval Analysis and Reliable Computing

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Abstract. The contribution of T. Sunaga to interval analysis and reliable computing is not well-known amongst specialists in the field. We present and comment Sunaga's basic ideas and results related to the properties of intervals and their application.

Keywords: History of mathematics, Interval analysis, Reliable computing

The interval concept is on the borderline linking pure mathematics with reality and pure analysis with applied analysis. T. Sunaga

1. Introduction

Interval analysis and reliable computing are interdisciplinary fields, combining both abstract mathematical theories and practical applications related to computer science, numerical analysis and mathematical modelling. At present there exist hundreds of related publications including many monographs, proceedings and collections of papers, more than twenty international meetings have been organized. It is a commonplace view that interval analysis will continue to play a significant role in applied mathematics, especially in mathematical modelling and reliable computing. Undoubtedly the development of interval analysis will become of interest to the history of mathematics. After four decades of intensive and successful development it is now time to look back and give tribute to the pioneering research in this field.

Some early (and rather premature) ideas about an interval calculus can be found in [30], [44]. Two papers can be considered as pioneering works in this field: one by the Japanese mathematician T. Sunaga [38], which summarizes the results of his Master Thesis accomplished in 1956 [36], and another by the Polish mathematician M. Warmus [42]. Both papers have been published almost at the same time and, apparently, completely independent of each other. Several years later a second paper appeared by Warmus [43], also the American mathematician R. Moore accomplished his dissertation in interval analysis [23] and published a monograph [24].

The early works on interval analysis published before the monograph [24] seem not to be well known. In the present article we shall briefly review and comment the work of T. Sunaga [38], which we shall refer to in the sequel as "Sunaga's paper". In some later publications (in Japanese) T. Sunaga uses interval analysis in problems related to mechanics and planning of production [40], [41]. This applied work will not be discussed here; however, in our opinion it also deserves special attention (and, possibly, translation into English). Sunaga's paper contains basic ideas about the algebraic properties of intervals and their application to numerical analysis; however, this work is rarely referenced in the literature.

In the illustrative examples we have done negligible changes in the notations: i) we use the familiar symbol $\{x \mid P(x)\}$ for "the set of elements x such that property $P(x)$ is satisfied" (Sunaga uses verbal expressions); ii) for inclusion we use the familiar notations $\alpha \in B$ or $A \subseteq B$, while Sunaga uses (for both cases) the same notation $\alpha \to B$, $A \to B$; iii) for the equality of intervals he uses the symbol $A \rightleftharpoons B$, which is associated with the definition: $A \rightleftharpoons B$, iff $A \to B$ and $A \leftarrow B$. It should be mentioned that the use of a single notation "$\to$" instead of both "$\in$" and "$\subseteq$" in numerical calculations is very convenient, because it allows us in the relation $\alpha \to B$ not to specify whether a number α is rounded (and thus is an interval so that we should write $\alpha \subseteq B$) or is exact (in which case we should write $\alpha \in B$); as a typical example see Example 8.2 in [38].

Finally let us mention, that Sunaga's paper is richly illustrated by 17 figures, which contributes to easy reading; and a great number of examples is provided. The final five examples demonstrate the application of interval analysis to reliable computing and present a special interest.

Our article is organized as follows. In Section 2 we briefly review Sunaga's theory of intervals as developed in Sections 1–7 in his work. In Section 3 we present and comment his ideas about the practical implementation of interval analysis to verified solution of numerical problems as given in Sections 8 and 9 of [38]. In Section 4 we make an attempt to briefly trace the further development of the field in relation to Sunaga's work. In the last Section we give a short biography of T. Sunaga.

2. Sunaga's "Theory of an Interval Algebra"

From the introduction of [38] we learn that T. Sunaga arrived to his "Theory of an Interval Algebra" while working in the field of Communication Theory [37]; it is mentioned that this latter work is related to C. Shannon's theory [35] and especially to issues of finiteness and discreteness. The work [26] is mentioned as an example of treating round-off errors by other methods.

Significance of interval (Section 1 of [38]). This introductory Section starts with an example of the numerical computation of $\sqrt{2}$ and points out that a real number is actually defined by a sequence of rational intervals, that is intervals with rational endpoints. Therefore "the concept of an interval is more fundamental than that of a real number".

If the rounding mode is known, then an interval may be presented by a single rational number. For example, in the calculation of $\sqrt{2}$, we obtain a sequence of rational numbers, such as 1, 1.4, 1.41, 1.414, ... The numbers in the sequence can be interpreted as intervals containing the value $\sqrt{2}$, i. e. the number 1.4 as the interval [1.4, 1.5], the number 1.41 as [1.41, 1.42] etc. Thus, in applications, a number is often understood as an interval. “To denote an interval, we need not necessarily use two rational numbers”. The shorthand notation $\langle\alpha\rangle$ is introduced and used throughout the paper to denote the tightest interval enclosing a decimal floating point number α, assuming that the number has been rounded to the nearest, e. g. $\langle 1.414\rangle = [1.4135, 1.4145]$. As may be seen from the sequel directed roundings are used systematically in many of the examples, which makes Sunaga's paper an early foregoer of the computer arithmetic approach, cf. e. g. [13].

Further in the introduction a clear distinction is made between the computational approach using intervals and the statistical approach. “The reader may think of such more familiar expressions as ‘*Statistically ...*’ or ‘*The statistical values are ...* ’. But probable or stochastical numerical values are not different from our physical quantities and should also be described by intervals.” The author summarizes the discussion of the practical importance of the interval concept by an original thought, which we have chosen as motto of the present article.

In [38] the type of letter used specifies the mathematical object. Small Greek letters $\alpha, \beta, ..., \gamma$, mean real numbers; intervals are denoted by roman letters, thereby upper case roman letters have different meaning than lower case roman letters: the letters $A, B, ..., C$, mean intervals in expressions involving interval arithmetic operations, whereas the letters $a, b, ..., c$, are used to denote variables or parameters of functions and functionals. These variables or parameters may have interval values, then instead of functions we have sets of functions or instead of functionals we have ranges of functions over interval parameters. Boldface roman letters $\mathbf{A}, \mathbf{B}, ..., \mathbf{C}$, denote interval vectors.

The interval lattice (Section 2 of [38]). In this second Section the inclusion partial ordering in the set of intervals and the induced lattice operations “join” (joined union, convex hull) $X \vee Y = \inf_{\subseteq}\{Z \mid X \subseteq Z, Y \subseteq Z\}$ and “meet” (intersection) $X \wedge Y = \sup_{\subseteq}\{Z \mid Z \subseteq X, Z \subseteq Y\}$ are discussed in some detail (typically, the definitions of join and meet are given in [38] in verbal form). Why the author starts his exposition with the inclusion relation, and not with the arithmetic operations? He obviously realizes the important role of the partial order in the set of intervals, but also he makes advantage of the fact that order relations belong to a well studied algebraic area — lattice theory [2]. Theorem 1 treats properties of the lattice operations and Theorem 2 states that the set of intervals together with the inclusion relation is a lattice. Another result (not formulated as a theorem) considers the inclusion isotonicity of lattice polynomials.

Let us note that the operation join is extensively used throughout the paper in the special case when the intervals are degenerate, that is numbers. The computational technique using the join of two numbers has been demonstrated in the examples given throughout [38] and in the formulation and proofs of some theorems (cf. e. g. Theorem 3).

The arithmetic operations (Section 3 of [38]). In this Section the arithmetic operations for intervals are introduced — to our knowledge for first time. Passing from the authors verbal form to symbolic notations we obtain:

Definition 6. Sum, difference, product and quotient of X, Y:
$X + Y = \{\xi + \eta \mid \xi \in X, \eta \in Y\}$;
$X - Y = \{\xi - \eta \mid \xi \in X, \eta \in Y\}$;
$XY = \{\xi\eta \mid \xi \in X, \eta \in Y\}$;
$X/Y = \{\xi/\eta \mid \xi \in X, \eta \in Y\}$, $0 \notin Y$.

Definition 7 introduces the operators $-X = 0 - X$ and $X^{-1} = 1/X$. Using these operators subtraction and division are reduced to addition, resp. multiplication by reciprocal, see Theorem 4. Further, Theorem 5 states that commutative and associative laws hold true for addition and multiplication. The last part of Theorem 5 contains the relation $X(Y + Z) \subseteq XY + XZ$, denoted later in the literature on interval analysis as "subdistributive law". It has been noted that the subdistributivity law implies the inclusion $(X - 1)X \subseteq X^2 - X$, and, more generally, that computation of the range of an algebraic polynomial "is better carried out by the Horner's method", symbolically, $(...((A_nX + A_{n-1})X + A_{n-2}X + ... + A_1)X + A_0 \subseteq A_nX^n + A_{n-1}X^{n-1} + ... + A_1X + A_0$. Theorem 6 gives the distributivity of addition, resp., multiplication, with respect to join. The proof supplied demonstrates the simultaneous use of arithmetic and lattice interval operations.

Theorems 7 and 8 contain the cancellation law for addition, resp. for multiplication. The algebraic solution of simple equations in interval arithmetic is discussed, e. g. the solution of $[1, 2] + X = [2, 5]$ is $X = [1, 3]$. To summarize, in this Section it is proved that the set of intervals together with addition (multiplication) is an abelian cancellative monoid (in case of mutiplication intervals containing 0 are excluded), moreover the subdistributive relation is established.

The use of join in presenting intervals via endpoints can be seen from the following example:

Example 3.3. We have

i) $(\alpha_1 \vee \alpha_2)(\beta_1 \vee \beta_2) = \alpha_1\beta_1 \vee \alpha_1\beta_2 \vee \alpha_2\beta_1 \vee \alpha_2\beta_2$
$= \min_{i,j}\{\alpha_i\beta_j\} \vee \max_{i,j}\{\alpha_i\beta_j\}$.

ii) $\langle 1.414 \rangle = 1.414 - 5 \times 10^{-4} \vee 1.414 + 5 \times 10^{-4}$
$= 1.414 + (-5 \times 10^{-4} \vee 5 \times 10^{-4}) = 1.414 + (-5 \vee 5) \times 10^{-4}$.

Note that the above notation avoids the necessity to indicate which end-point of a given interval is left and which one is right (in fact this is often unknown in advance — i. e. when the endpoints are represented by some expressions that have to be evaluated).

Multidimensional intervals (Section 4 of [38]). In this Section the definitions of the arithmetic operations (addition, subtraction, multiplication by scalar) and the lattice operations are extended to n-dimensional intervals of the form $(X_1, ..., X_n)$, where $X_1, ..., X_n$ are usual (one-dimensional) intervals.. A useful discussion related to multiplication by "interval scalar" is presented. Multiplication of an n-

dimensional interval by a real scalar as induced by addition of n-dimensional intervals is: $x(X_1, ..., X_n) = (xX_1, ..., xX_n)$. According to Sunaga's general methodology, multiplication by interval scalar should be the extension of the above definition for $x \in X$, that is:

$$X(X_1, ..., X_n) = \{(xX_1, ..., xX_n) \mid x \in X\}.$$

Sunaga gives the following subdistributive relation (see formula 4.6 in [38]):

$$X(X_1, ..., X_n) \subseteq (XX_1, ..., XX_n).$$

It is noted that the relation $X(X_1, ..., X_n) = (XX_1, ..., XX_n)$ does not always hold; and the following simple example for the expressions $X(1, U)$ and (X, XU) for given intervals X, U is discussed:

Example 16. $X(1, U) \subseteq (X, XU)$. Let $X = 1 \vee 2$, $U = 1.0 \vee 1.2$, then $X(1, U) = (1 \vee 2)(1, 1.0 \vee 1.2)$; $(X, XU) = (1 \vee 2, 1.0 \vee 2.4)$.

The "wrapping effect" is demonstrated and visualized (cf. Fig. 8 in [38]). A connection is made to similar expressions in the one-dimensional case, e. g. the expressions $X(1 + U)$ and $X + XU$. The influence of the errors in the coefficients of a system of linear equations on the solutions of the system is discussed at the end of the Section.

Interval functions and functionals (Section 5 of [38]). Note that the value of an expression involving multiplication by interval scalar is not an interval in general. This viewpoint is further extended in the definition of interval function. The value of an interval function is not an interval. According to Sunaga's definition, an interval function is a set of real functions, depending on parameters, taking values from given intervals. For example, the set of functions $f(\,\cdot\,; a) = \{f(\,\cdot\,; \alpha) \mid \alpha \in a\}$ is an interval function. Here a may be a n-dimensional interval, e. g. for $n = 2$ we have: $f(\,\cdot\,; a, b) = \{f(\,\cdot\,; \alpha, \beta) \mid \alpha \in a, \beta \in b\}$. An interval functional is a special case of an interval function, when the function depends only on parameter(s). Hence an interval functional is the range of a function over a given (n-dimensional) interval. For instance, if $f(\xi)$ is a function defined on some domain, and a be an interval from this domain, then the set $f(a) = \{f(\alpha) \mid \alpha \in a\}$, is an interval functional.

Interval functionals (and functions) involve intervals as arguments; the latter are denoted in the same way as real variables — by lower case Roman letters. For comparison, the intervals involved in an interval-arithmetic expression are denoted by upper case letters. The difference between the two concepts is demonstrated by examples: we have $a(a + b) = a + ab$, but $A(A + B) \neq A + AB$. Another example:

$$x(x - 1) = \{\xi(\xi - 1) \mid \xi \in x\} \subseteq X(X - 1) \subseteq X^2 - X. \tag{1}$$

Such brief notation of an interval functional is very useful for numerical analysis, where a variable x in expressions like $x(x - 1)$ may easily turn from a real number to an interval depending on whether x is machine representable or not.

The sharp difference made between ranges of functions and interval arithmetic expressions outlines the subordinate role of interval arithmetic as a tool for numerical computation or enclosure of functional ranges. Thus one of the main objectives of interval analysis is clearly formulated in this Section.

Example (1) expresses the simple rule that the range of a function is contained in the corresponding interval arithmetic expression obtained by formally replacing the real arithmetic operations by interval-arithmetic ones. This rule is not formulated in general. There may be at least two reasons for this: first, the general style of the paper is to present the new ideas by simple examples, second, the author may have considered this rule as a too obvious consequence from the definitions of the interval-arithmetic operations. However, it should be mentioned that this rule has been used at several places in the paper on different expressions of the form $f(x, y, ..., z)$. Using Sunaga's notation of an interval functional this rule may be formulated symbolically as:

$$f(x, y, ..., z) \subseteq F(X, Y, ..., Z),\ x \in X, y \in Y, ..., z \in Z, \tag{2}$$

where F means the interval arithmetic expression obtained by formally replacing the variables of f by the respective intervals and the real arithmetic operations by respective interval-arithmetic ones.

It may be also of some interest to note that in this Section the author uses the concept of functional, as used in the theory of distributions [34], e. g. the Dirac δ function is a functional, which is extended for interval arguments (and the same for its derivative).

Differentiation (Section 6 of [38]). Derivatives are defined in the spirit of the previous Section 5. Thus, if f is a differentiable function on some domain and a is an interval from this domain, then $f'(a)$ means the range of the functionf' in the interval a, that is, $f'(a) = \{f'(\alpha) \mid \alpha \in a\}$, and is called the *differential coefficient of f on the interval a*. The following two propositions are formulated in this section (numbered as formulae 6.4 and 6.5 in Sunaga's paper).

i) If x, x_1 and dx are intervals such that $x_1 \subseteq x$, $x_1 + dx \subseteq x$, and x is from the definition domain of f, then $f(x_1 + dx) \subseteq f(x_1) + f'(x)dx$.

ii) If x, x_1, x_{11} and dx are intervals, such that $x_{11} \subseteq x_1 \subseteq x$, $x_{11} + dx \subseteq x$, then $f(x_{11} + dx) \subseteq f(x_{11}) + f'(x_1)dx + (1/2)f''(x)(dx)^2$.

The first proposition is an interval Taylor formula of first order (interval mean-value theorem) and the second proposition is an interval Taylor formula of second order. The interval mean-value theorem has been used several times in the applications, see Section 9. The formula is often used in interval analysis in the special case when x_1 and dx are numbers and (only) x is an interval. Sunaga has found a rather general form of the formula. The assumed inclusions of the intervals involved in the second order formula are visualized, see Fig. 12 in [38].

The topological background (Section 7 of [38]). In this Section the author looks for already studied algebraic systems, which incorporate the partial order relation inclusion together with arithmetic operations. He thus arrives to analogies between interval systems and topological groups [31]). The idea is to consider intervals as neighbourhoods of numbers and to apply the theory of topological spaces. The inclusion relation plays a basic role in such considerations.

Sunaga gives the definition of a topological group and discusses it's properties in some detail. At the end of the section it is suggested that the set of neighbourhoods, i. e. intervals, is completed into a group, so that inverse elements exist. In this section there are no particular original results, except for the general proposal to use tools from topological algebra for the study of the interval system.

3. Sunaga's Proposal for Application of Interval Analysis to Reliable Numerical Computation

In the last two sections of [38] practical application of interval calculus to reliable numerical computation are proposed.

Practical interval calculus (Section 8 of [38]). This Section is devoted to the midpoint-radius (center-radius) presentation of intervals. Such presentation is indeed of extraordinary importance in the practical applications. As seen from Example 8.1 several different notations are used for the center-radius form $A = (\alpha, \alpha_0) = \alpha + (-\alpha_0 \vee \alpha_0)$, $\alpha_0 \geq 0$:

Example 8.1. $1 \pm 0.01 = (1.00, 10^{-2}) = 1.00; 1$

$$\langle 1.414 \rangle = (1.4140, 5 \times 10^{-4}) = 1.4140; 5.$$

In this example the author introduces another brief notation for intervals in center-radius form: for the interval with center 1.00 and radius 0.01 he writes simply 1.00;1, the interval with center 1.4140 and radius 0.0005 is denoted $1.4140; 5$ (which was also denoted $\langle 1.414 \rangle$).

The author gives formulae for the arithmetic operations in center-radius form:

Theorem 11. Let $A = (\alpha, \alpha_0) = \alpha + (-\alpha_0 \vee \alpha_0)$, $\alpha_0 \geq 0$, $B = (\beta, \beta_0) = \beta + (-\beta_0 \vee \beta_0)$, $\beta_0 \geq 0$, be two intervals. We have:
i) $(\alpha, \alpha_0) + (\beta, \beta_0) = (\alpha + \beta, \alpha_0 + \beta_0)$;
ii) $(\alpha, \alpha_0) - (\beta, \beta_0) = (\alpha - \beta, \alpha_0 + \beta_0)$;
iii) $(\alpha, \alpha_0)(\beta, \beta_0) = (\alpha\beta + \alpha_0\beta_0, \alpha_0\beta + \alpha\beta_0)$, $\alpha \geq \alpha_0 \geq 0$, $\beta \geq \beta_0 \geq 0$;

$$\text{iv)}\ \frac{(\alpha, \alpha_0)}{(\beta, \beta_0)} = \left(\frac{\alpha\beta + \alpha_0\beta_0}{\beta^2 - \beta_0^2}, \frac{\alpha_0\beta + \alpha\beta_0}{\beta^2 - \beta_0^2}\right),\ \alpha \geq \alpha_0 \geq 0, \beta > \beta_0 \geq 0.$$

Sunaga notes that these "operations are similar to those of complex numbers":

$(\xi + i\xi_0) + (\eta + i\eta_0) = (\xi + \eta) + i(\xi_0 + \eta_0)$,
$(\xi + i\xi_0) - (\eta + i\eta_0) = (\xi - \eta) + i(\xi_0 - \eta_0)$,
$(\xi + i\xi_0)(\eta + i\eta_0) = (\xi\eta - \xi_0\eta_0) + i(\xi_0\eta + \xi\eta_0)$,
$(\xi + i\xi_0)/(\eta + i\eta_0) = (\xi\eta + \xi_0\eta_0)/(\eta^2 + \eta_0^2) + i(\xi_0\eta + \xi\eta_0)/(\eta^2 + \eta_0^2)$.

It has been also noticed that the product and the quotient of two intervals A, B do not have as centers the product, resp. the quotient, of the centers of A and B. Sunaga gives formulae for the tightest interval with center the quotient of A, B, which includes the interval-arithmetic quotient of the intervals A, B:

Theorem 12. If $\alpha \geq \alpha_0 \geq 0$, and $\beta > \beta_0 \geq 0$, then

$$\frac{(\alpha, \alpha_0)}{(\beta, \beta_0)} \subseteq (\frac{\alpha}{\beta}, \frac{\alpha_0 + (\alpha/\beta)\beta_0}{\beta - \beta_0}). \tag{3}$$

Further, the following important observation is made: "In practical calculations, it is often meaningless to calculate $\alpha\beta$ or α/β accurately and numerals should be rounded adequately. In such cases the following theorem is useful:

Theorem 13. If $A = (\alpha, \alpha_0)$, $\alpha \in (\alpha', \alpha'_0)$, then $A \subseteq (\alpha', \alpha_0 + \alpha'_0)$."

Example 8.2. We have $1.432 \in 1.43; 14$, hence $1.432; 50 \subseteq 1.43; 6$.

A result for multiplication analogous to Theorem 12 is given using the above result (see Example 8.2, part ii):

If $\alpha \geq \alpha_0 \geq 0$, and $\beta \geq \beta_0 \geq 0$, then

$$(\alpha, \alpha_0)(\beta, \beta_0) \subseteq (\alpha\beta,\ \alpha_0\beta_0 + \alpha_0\beta + \alpha\beta_0). \tag{4}$$

For practical computations the center-radius form has the advantage that the significant digits are not repeated twice (as in the end-point form, cf. e. g. the interval $[1.4135, 1.4145]$, where the digits 1.41 are repeated twice. The arithmetic rules using center-radius form (Theorem 11) and notation introduced in Example 8.1 (like $1.00; 1$) are demonstrated in some detail in Example 8.3 in [38]:

Example 8.3. i) Addition:
$1.689; 4 + 2.745; 1 = 4.434; 5$,
$3.624; 8 + 1.24; 3 = 4.864; 38 \subseteq 4.86; (0.4 + 3.8) \subseteq 4.86; 5$.

ii) Subtraction:
$3.429; 5 - 1.201; 2 = 2.228; 7$,
$6.724; 7 - 2.30; 4 = 4.424; 47 \subseteq 4.42; 6$.

iii) Multiplication:
$(0.4320; 5)(0.3810; 5) \subseteq 0.16459; 43 \subseteq 0.1646; 5$. ...

vi) $\frac{7\sqrt{2}-\pi\sqrt{3}}{\pi^2+\sqrt{3}} \in \frac{7\times 1.4142140;5-3.1415930;5\times 1.7320510;5}{(3.1415930;5)^2+1.7320510;5}$

$\subseteq \frac{9.899498;4-5.441399;3}{9.869607;4+1.732051;1} \subseteq \frac{4.458099;7}{11.601658;5} \subseteq 0.384264;1.$

These examples demonstrate the calculation of guaranteed enclosure of rational expressions — an approach known as "naive interval arithmetic".

Examples of numerical calculations (Section 9 of [38]). This Section contains a proposal for an application of interval analysis to reliable numerical computation (on digital computers). The form of the presentation is a detailed discussion of five case studies, which we shall next comment in some detail.

Example 9.1. This case study is devoted to the solution of nonlinear equations by means of a proposed Newton-like interval method. The author describes the method on a particular problem, namely the equation $f(x) \equiv x^3 - 3x + 1 = 0$ as follows. It is assumed first that we know an interval X_0 containing a solution. The method produces a sequence of intervals $X_1, X_2, ...$

We have $f(0.3) = -f(0.4)$, hence there exists a solution $\alpha \in [0.3, 0.4] = X_0$. The function value at the middle point $\mathrm{mid}(X_0) = 0.35$ is $f(0.35) = -0.007125$. We have $f'(x) = 3(x^2-1)$, therefore $f'(X_0) = f'(0.3 \vee 0.4) \subseteq 3([0.3, 0.4]^2-1) \subseteq -[2.52, 2.73]$. Denoting $dx = \alpha - \mathrm{mid}(X_0)$ we obtain

$$0 = f(\alpha) = f(\mathrm{mid}(X_0) + dx) = f(\mathrm{mid}(X_0)) + f'(\mathrm{mid}(X_0) + \theta dx)dx,$$

where $0 \leq \theta \leq 1$. Solving for dx we get (using outward rounding!)

$$dx = -\frac{f(\mathrm{mid}(X_0))}{f'(\mathrm{mid}(X_0)+\theta dx)} \in -\frac{f(\mathrm{mid}(X_0))}{f'(X_0)}$$

$$= -\frac{0.007125}{[2.52, 2.73]} \subseteq -[0.0026, 0.0029].$$

Therefore

$$\alpha = \mathrm{mid}(X_0) + dx \in \mathrm{mid}(X_0) - \frac{f(\mathrm{mid}(X_0))}{f'(X_0)}$$

$$= 0.35 - [0.0026, 0.0029] = [0.3471, 0.3474] = X_1.$$

To formulate the above method symbolically, assume that X_0 contains a zero of $f(x) = 0$. The next interval X_1 containing the solution is computed using outward computer arithmetic by means of the formula:

$$X_1 = \mathrm{mid}(X_0) - f(\mathrm{mid}(X_0))/f'(X_0),$$

where $\mathrm{mid}(X_0)$ is the middle point of X_0 and $f'(X_0)$ is the range of the function f' on X_0. The procedure can be continued, that is we have

$$X_{n+1} = \mathrm{mid}(X_n) - f(\mathrm{mid}(X_n))/f'(X_n),\ n = 0, 1, 2, ... \qquad (5)$$

until the needed accuracy has been achieved. From the derivation of the method it becomes clear, that instead of the midpoint any other point γ of the interval X_0 can be used; and this is actually done in the next example, where instead of $\gamma = \mathrm{mid}(X_0)$ the left end-point of X_0 is taken (due to the fact that a table value is known only for the end-points).

Let us note that during the computations (here and in other examples below) the author makes use of the rule (1) saying that the range of a function is contained in the corresponding interval arithmetic expression obtained by formally replacing the real arithmetic operations by interval-arithmetic ones. For example, the range of the function $f'(x) = 3(x^2-1)$ over the interval $X_0 = [0.3, 0.4]$ is computed as follows: $f'(X_0) = 3(x^2-1)\mid_{x\in X_0} \subseteq 3(X_0^2-1) = 3([0.3,0.4]^2-1) \subseteq -[2.52, 2.73]$, or in Sunaga's notation $f'(0.3 \vee 0.4) \to 3((0.3\vee 0.4)^2-1) \to -(2.52 \vee 2.73)$.

Example 9.2. The author considers one more example for the reliable solution of nonlinear equations. Here the method described in Example 9.1 is implemented to the equation $f(x) = xe^x - 2\sin x$ with initial interval $X_0 = [0.62, 0.63]$.

It is assumed that table values for e^x, $\sin x$ and $\cos x$ at the endpoints of the interval X_0 are known approximately, namely:

$e^{0.620} \in \langle 1.8589280\rangle$, $e^{0.630} \in \langle 1.8776107\rangle$, $\sin 0.620 \in \langle 0.5810352\rangle$,
$\sin 0.630 \in \langle 0.5891447\rangle$, $\cos 0.620 \in \langle 0.81387\rangle$, $\cos 0.630 \in \langle 0.80815\rangle$.

Since there is no table value for $f(0.625)$, that is at the midpoint of the interval X_0, the method of Example 9.1. is considered in the form $X_1 = \gamma - f(\gamma)/f'(0.62\vee 0.63)$, $\gamma \in X_0$, and γ is taken to be the left endpoint of X_0 (instead of midpoint), $\gamma = 0.62$. Then the first approximation X_1 is calculated (using outward roundings) to be $X_1 = [0.6265, 0.6270]$. The computations are performed as follows:

$$f(0.62) = 0.62e^{0.62} - 2\sin 0.62 \in 0.62 \times 1.858928;1 - 2\times 0.581035;1$$
$$\subseteq 1.1525354;7 - 1.162070;2 \subseteq -0.009535;9;$$

$$f'(0.62\vee 0.63) \subseteq (1+(0.62\vee 0.63))e^{0.62\vee 0.63} - 2\cos(0.62\vee 0.63)$$
$$\subseteq (1.62\vee 1.63)(1.858\vee 1.878) - 2(0.808\vee 0.814)$$
$$\subseteq (3.009\vee 3.062) - (1.616\vee 1.628) \subseteq 1.381\vee 1.446;$$

$$dx \in -\frac{f(0.62)}{f'(0.62\vee 0.63)} \subseteq \frac{(9.526\vee 9.544)\times 10^{-3}}{1.381\vee 1.446} \subseteq (6.58\vee 6.91)\times 10^{-3}.$$

Let α be the required root, then $\alpha = 0.62 + dx \in 0.6265 \vee 0.6270 = X_1$. The next step is to proceed in a similar manner, calculating

$$dx \in -f(0.6267)/f'(0.6265\vee 0.6270).$$

This problem differs from the previous one in the way of computation of the values of the function and its derivative at the end-points of the intervals; here we cannot obtain exact values as in Example 9.1, because we are given inexact table values for the functions exp x, sin x and cos x, which must be outwardly rounded; for instance: exp $0.620 \in \langle 1.18589280\rangle = 1.858928;1$. Now, to repeat the process, we need to compute $f(\text{mid}(X_1)) = f(0.6267)$, but such a value is not available in the table used. Hence, this example motivates the need of reliable methods for interpolation, which are discussed in the next example.

Example 9.3. In this example the problem of reliable interpolation of functional values with prescribed errors is considered. It is to be noted that in the today's literature of interval analysis there exist many methods devoted to interpolation of points in the plane with given errors in the y-coordinates, without an assumption that the points lie on the graph of a certain smooth function. Here the author considers namely the latter formulation. His method is again based on the interval mean-value theorem and has been demonstrated on the reliable interpolation of nonexact (table) functional values of the function exp x. We give below the arguments of the author.

"Interval relations such as
$e^{0.620} \in 1.85892800; 5,\ \ e^{0.621} \in 1.86078790; 5$
can be used to evaluate $e^{0.620+\nu}$, $\nu \in 0.000 \vee 0.001$, as follows. Using that $f(x_1 + dx) \in f(x_1) + f'(x)dx$, where $x_1 \in x$, $x_1 + dx \in x$, then, by means of the relation

$$e^{0.620+\nu} \in e^{0.620} + e^{0.620 \vee 0.621}\nu,$$

we proceed to obtain a sufficiently accurate value. For instance, when $\nu = 0.0004$, we have:

$$\begin{aligned} e^{0.6204} &\in 1.85892800; 5 + (1.8589 \vee 1.8608)0.0004 \\ &\subseteq 1.85892800; 5 + (1.8599; 10)4 \times 10^{-4} \\ &\subseteq 1.85892800; 5 + 7.4396; 40 \times 10^{-4} \subseteq 1.85967196; 45. \end{aligned}$$

Thus, we have seen that making use of differential coefficients, we can increase the accuracy of the interpolation".

Example 9.4. This Example is devoted to the reliable computation of definite integrals; as a particular case study the computation of the definite integral $\int_0^1 1/(1 + x^2)dx$ by a modification of Simpson's rule is considered. The author's idea is to enclose the remainder term by carefully computing the range of the fourth derivative of the subintegral function (using directed floating-point arithmetic) over each subinterval of the mesh. The idea can be easily extended to a large class of formulae for numerical quadratures. Below we reproduce the calculations from [38]:

"Simpson's rule is as follows:

$$\int_{-h}^{h} f(x)dx \in \frac{h}{3}\left(f(h) + f(-h) + 4f(0)\right) - \frac{h^5}{90} f^{(4)}(-h \vee h),$$

hence it can be written

$$\int_a^b f(x)dx \in \frac{h}{3}\left(y_0 + y_{2n} + 2(y_2 + ... + y_{2n-2}) + 4(y_1 + ... + y_{2n-1})\right)$$

$$-\frac{h^5}{90}\sum_{i=1}^{n} f^{(4)}\left(x_{2i-1} + (-h \vee h)\right),$$

where $h = (b-a)/2n$. For instance, to integrate numerically

$$\int_0^1 \frac{dx}{1+x^2} \left(= \frac{\pi}{4}\right)$$

one proceeds as follows

$$f(x) = \frac{1}{1+x},\ f^{(4)}(x) = \frac{4!(1-x^2(10-5x^2))}{(1+x^2)^5},\ \frac{f^{(4)}(0 \vee 0.2)}{4!} \subseteq 1 \vee 0.49.$$

Similarly,

$$\frac{f^{(4)}(0.2 \vee 0.4)}{4!} \subseteq (-0.48) \vee 0.54,\ \frac{f^{(4)}(0.4 \vee 0.6)}{4!} \subseteq -(0.06 \vee 1.15),$$
$$\frac{f^{(4)}(0.6 \vee 0.8)}{4!} \subseteq -(0.11 \vee 0.94),\ \frac{f^{(4)}(0.8 \vee 1)}{4!} \subseteq -(0.06 \vee 0.49).$$

Then

$$\frac{1}{4!}(f^{(4)}(0 \vee 0.2) + f^{(4)}(0.2 \vee 0.4) + f^{(4)}(0.4 \vee 0.6)$$
$$+f^{(4)}(0.6 \vee 0.8) + f^{(4)}(0.8 \vee 1)) \subseteq -2.57 \vee 1.31.$$

Hence

$$\int_0^1 \frac{dx}{1+x^2} \in \frac{0.1}{3}(1 + \frac{1}{2} + 2(\frac{1}{1.04} + \frac{1}{1.16} + \frac{1}{1.36} + \frac{1}{1.64})$$
$$+4((\frac{1}{1.01} + \frac{1}{1.09} + \frac{1}{1.25} + \frac{1}{1.49} + \frac{1}{1.81})) - \frac{(0.1)^5 4!}{90}(-2.6 \vee 1.4).$$

Finally one gets $\pi \in 3.141593; 2$".

Example 9.5. In this Example the IVP for the ODE: $\eta' = f(\xi, \eta)$, $\eta(\xi_0) = y_0$ is considered; the value y_0 can be an interval one. Here we slightly change the original notation to make a difference between numerical (real) values ξ, η and interval values x, y; in [38] all values are denoted using the letters x, y, which means that they may be intervals — in consistence with Sunaga's methodology. The author's method can be reformulated as follows. Consider a mesh: $\xi_0, \xi_1, \xi_2, \ldots$. Denote the range of η over a subinterval $x_k = \xi_k \vee \xi_{k+1}$ by $y_{k+1} = \eta(x_k)$, resp. the range of η' over x_k by $y'_k = \eta'(x_k)$. Then, using the interval mean-value theorem, we have:

$$\eta'_1 \in y'_1 = f(x_0, y_0),\ \eta_1 \in y_1 = y_0 + f(x_0, y_0)(\xi_1 - \xi_0);$$
$$\eta'_2 \in y'_2 = f(x_1, y_1),\ \eta_1 \in y_2 = y_1 + f(x_1, y_1)(\xi_2 - \xi_1).$$

The author points out that the accuracy can be increased by higher order derivatives. Thus the foundation of a large class of methods for the verified solution of IVP for ODE, using interval mean-value theorem, is given. Using the example: $y' = 2 - x/y,\ \ y(0) = 1$, he shows the technique of obtaining safe enclosures by special roundings and of increasing the accuracy by using higher order derivatives. His calculations are as follows:

"The x-axis is divided into $x_0 = 0 \vee 0.1,\ x_1 = 0.1 \vee 0.2, \ldots.$ Differential coefficients of high order are calculated from the relations $yy' = 2y - x,\ yy'' + {y'}^2 = 2y' - 1, \ldots.$

Denote $y_0 = y(0 \vee 0.1),\ y_1 = y(0.1 \vee 0.2), \ldots.$ We then have

$y_0' \in 2 - \frac{(0 \vee 0.1)}{1} \subseteq 1.9 \vee 2,\ y_1 \in 1 + (1.9 \vee 2)x.$

To raise its accuracy we must first calculate the differential coefficient of the second order, i. e.

$y_0'' \in \frac{y_0(2-y_0')-1}{y_0} \subseteq \ldots \subseteq -(0.66 \vee 1).$

Since

$y' \mid_{x=0} = 2,\ y_0 \subseteq 1 + 2x - \frac{0.66 \vee 1}{2}x^2;$

therefore

$y(0.1) \in 1 + 0.2 - (0.33 \vee 0.5) \times 10^{-2} \subseteq 1.1957; 9,$
$y'(0.1) \in 2 - (0.66 \vee 1)0.1 \subseteq 1.917; 17.$

Next we go to the second interval $0.1 \vee 0.2$. In this interval we have

$y_1' \in 2 \vee \left(2 - \frac{0.2}{1.19}\right) \subseteq 2 \vee 1.83,$
$y_1 \in 1.19 \vee 1.20 + (2 \vee 1.83)(x - 0.1),$
$y(0.2) \in 1.19 \vee 1.20 + (2 \vee 1.83)0.1 \subseteq 1.37 \vee 1.40.$

Using this value we can evaluate more accurately, i. e.,

$y_1' \in 2 - \frac{0.1 \vee 0.2}{1.19 \vee 1.40} \subseteq 2 - (0.07 \vee 0.17) \subseteq 1.83 \vee 1.93.$

Hence

$$y_1'' \in \frac{y_1'(2-y_0')-1}{y_1} \subseteq \frac{(1.83 \vee 1.93)(0.07 \vee 0.17)-1}{1.19 \vee 1.40} \subseteq -\left(\frac{0.67 \vee 0.88}{1.19 \vee 1.40}\right) \subseteq -(0.47 \vee 0.74);$$

$$y_1 \in y(0.1) + y'(0.1)(x - 0.1) + \frac{y_1''}{2}(x - 0.1)^2 \subseteq 1.1957; 9 + 1.917; 17(x - 0.1) - \frac{(0.47 \vee 0.74)}{2}(x - 0.1)^2;$$

$y(0.2) \in 1.1957; 9 + 0.1917; 17 - (30; 7) \times 10^{-4} \subseteq 1.3844; 33.$

As stated above, we can integrate in the interval $0.1 \vee 0.2$ as accurately as we wish and carry it out over a wide region. We cut the x-axis into intervals, and reconnect them again".

4. Sunaga's Work and Later Research

Let us mention some later investigations in the field of interval analysis that are related to Sunaga's paper, and some known to us occasions, when his ideas have been exploited and further developed.

The join notation. The presentation of an interval as a "join" of its end-points has been used to denote inner arithmetic operations and to perform effective computations with intervals [14]. Recall that the inner difference $A -^- B$ is the algebraic solution of $B+X = A$ or of $A-X = B$ (at least one of these equations has a solution, and if both are solvable, then the solutions coincide). Note that $X = A-^- B$ always exists, therefore it defines an operation, called *inner subtraction*; this algebraically induced operation is useful for the treatment of interval-arithmetic equations and ranges of functions. Therefore it is important to have simple presentation of this operation in terms of interval end-points. The inner difference of the intervals $A = [\alpha_1, \alpha_2]$ and $B = [\beta_1, \beta_2]$ is the interval with end-points $\alpha_1 - \beta_1$ and $\alpha_2 - \beta_2$. We have $\alpha_1 - \beta_1 \leq \alpha_2 - \beta_2$ or $\alpha_1 - \beta_1 > \alpha_2 - \beta_2$ depending on the widths w of the intervals A and B, that is we have

$$A -^- B = \begin{cases} [\alpha_1 - \beta_1, \alpha_2 - \beta_2], & \text{if } w(A) \geq w(B), \\ [\alpha_2 - \beta_2, \alpha_1 - \beta_1], & \text{if } w(A) < w(B). \end{cases}$$

Using Sunaga's notation we can simply write $A -^- B = (\alpha_1 - \beta_1) \vee (\alpha_2 - \beta_2)$. Such notation is used for the rest of the inner operations (addition, multiplication and division) and is very useful for performing symbolic manipulations. Similar symbolic notations have been further elaborated in [4], [17] and have contributed much for the algorithmic implementation of the interval arithmetic (especially interval multiplication) in computer algebra systems [32].

Interval functions, differentiation. The sharp difference made by Sunaga between ranges of functions and interval arithmetic expressions outlines the subordinate role of interval arithmetic as a formal algebraic tool for the numerical computation or enclosure of ranges. Thus one of the main objectives of interval analysis is clearly formulated in his sections devoted to interval functions and differentiation. Using such methodology theorems for the presentation of ranges of monotone functions have been developed (see, e. g. [3], [16]). Furthermore, Sunaga's derivatives have been expressed in terms of inner interval arithmetic operations, see e. g. [15]; such expressions have been used to formulate efficient Newton-like methods (see, e. g. [5], [14]). The interval mean-value theorem and its application to numerical analysis is probably one of the most important Sunaga's discoveries.

The center-radius form. The analogies of the interval arithmetic operations in center-radius form with complex arithmetic noted by Sunaga are quoted in [1]. Such analogies are investigated by E. Kaucher [11] and more recently by V. Zyuzin [45]. A discussion of formulae related to (3) and (4) and their application can be found in [25], where it is noted that these formulae are special case of the complex interval arithmetic using discs, considered by P. Henrici [6]. A study of the algebraic properties of these operations and their relation to stochastic arithmetic has been undertaken in [20].

The topological background. In his section about the topological background of interval analysis Sunaga comes to the idea of algebraic embedding of intervals in a system with group properties. It should be noted that M. Warmus [42], [43] also proposes algebraic extensions. As it is well known, such extensions have been investigated in detail about a decade later — the isomorphic algebraic extensions of the order relation and the arithmetic operations are studied by E. Kaucher [10], [11]. Other important studies of abstract algebraic systems related to intervals, and more generally to convex bodies, have been performed by O. Mayer [21], H. Ratschek and G. Schröder [33] etc. The investigation of the properties of the interval-arithmetic operations by algebraic isomorphic extensions is of considerable interest for the solution of algebraic problems involving intervals. Recent results in this direction (some of them related to the center-radius presentation of intervals) are reported in [18], [19].

Applications. Two of the tools found by Sunaga are very important for applications: i) the arithmetic in center-radius form together with formulae for rounding and inclusion, and ii) the interval mean-value theorem and its application to verification algorithms, such as the Newton-like method for nonlinear equations (5), introduced by Sunaga in Example 9.1. In relation to this method it should be noted, that the author does not discuss conditions for convergence and contraction of the sequence of intervals, etc. Similar questions have been later studied by R. Moore, who introduces the notion of finite and global convergence [24] (the Newton-like method of R. Moore differs from (5) by intersecting each new interval by the previous one to assure that the new interval is nested in the previous one). In his chapter on interval arithmetic in [24] R. Moore mentions the work [38] in his references. However, a number of Sunaga's important results are not mentioned there, such as the cancellation law, the center-radius form of the interval arithmetic operations, etc.

Sunaga's method described in Example 9.1 is a foregoer of Newton-like interval methods for solving nonlinear equations, proposed later. Sunaga seems to be the first who uses outwardly directed roundings so that the bounding intervals provide guaranteed enclosures of the solution. In the methods formulated in [5], [14], intersection with previous intervals has been avoided by means of a suitably chosen interval-arithmetic expression for the range of the Newton operator much in the lines of Sunaga's definition of interval function.

The discussion of the relation between the interval approach and the statistical one in Section 1 of [38] deserves attention. It seems to us that this author's view has been later developed in relation to parameter identification, cf. e. g. [22].

Sunaga's idea to use interval analysis for the construction of numerical methods with verification, made interval analysis an important tool in mathematical modelling in applied sciences, such as mechanical engineering [40], [41]. Another research area where interval analysis finds application is electrical engineering (see, e. g. [12], [27], [28], [29]).

(*PHOTO*)

A photograph of Teruo Sunaga

5. A Brief Biography of T. Sunaga

- Teruo Sunaga is born April 20, 1929 in Tokyo.
- In March 1954 he graduates from the Department of Applied Mathematics in the Faculty of Engineering of the University of Tokyo and in 1956 he graduates from a master course [36].
- In March 1959 he graduates from the Doctor course of the Graduate School of Mathematical Physics of University of Tokyo.
- On April 1, 1959 he is appointed assistant professor at the Department of Mechanical Engineering for Production of University of Tokyo and on June 16 the same year he becomes associate professor at the same department.
- In March 1961 he becomes Doctor of Engineering from the University of Tokyo [39].

- On April 1, 1974 he becomes Professor at the Department of Mechanical Engineering for Production of University of Tokyo.
- From April 1, 1976 to March 31, 1978 he is Director of the University Computation Center.
- March 31, 1993 retires.
- May 7, 1993 becomes Professor Emeritus at Kyushu University.
- Prof. Teruo Sunaga died February 25, 1995.

6. Conclusion

The contribution of T. Sunaga to interval analysis and reliable computing is original and outstanding. He formulates and investigates the basic algebraic properties of the interval arithmetic operations together with the inclusion relation on a firm algebraic foundation. He formulates for the first time an interval form of Taylor's formula (the interval mean-value theorem and a formula of second order) and the center-radius form of the interval-arithmetic operations together with the formulae for "centered" multiplication and division. He demonstrates on a number of case studies how such tools can be applied for reliable numerical computation. He also shows the role of directed roundings for the safe numerical computation. In the Conclusion of his work the author gives a brief, clear and modest summary of his achievements as follows: *"An interval calculus is established algebraically from the lattice theoretical point of view so that it can be applied conveniently to the numerical calculation. Interval functions and functionals, and their differentiation are investigated and used effectively in some examples of applied analysis".*

T. Sunaga is applied mathematician (by education); he works in diverse applicational areas like communication theory, mechanical engineering and planning of production [37]–[41]. In his applied work he makes a systematic use of interval analysis, which, in our opinion, deserves to be studied.

In [7] (see also [8], [9]) Prof. M. Iri writes: *Sunaga considered all computational procedures, which had been traditionally defined on real numbers, as being too ideal and proposed to replace them by the procedures on real intervals in order to make everything "more realistic". His ideas were certainly influenced by the concepts from topological algebra. ... Sunaga studied many different kinds of numerical procedures including the Taylor-series interval solution of the initial-value problem of ordinary differential equations.*

It is also interesting to read Sunaga's vision about "future problems":

- *To investigate problems of numerical calculation connected with higher dimensional mathematics, for instance, matrix inversion, partial differential equations, etc.,*
- *To investigate direct applications of the interval calculus to physical and engineering problems,*

- *To revise the structure of the automatic digital computer from the standpoint of interval calculus and topology,*
- *To prove the applicability to other fields, of our view that scientific laws should be stated essentially in the language of finite elements and discrete topology.*

It seems that a lot of this ambitious program has been fulfilled, but also a lot remains to be done.

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