# COMPUTATION OF ALGEBRAIC SOLUTIONS TO INTERVAL SYSTEMS VIA SYSTEMS OF COORDINATES

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**Abstract** It is proved that (set-theoretic) interval multiplication is inverse inclusion isotone. The centered outward interval multiplication (co-multiplication) is studied in some detail with respect to inclusion isotonicity. To a system of linear interval algebraic equations we associate a system involving co-multiplication. The latter reduces to two real linear systems of the same size for the midpoint-radius coordinates of the unknown intervals. We show that under certain assumptions these real linear systems produce an inner inclusion for the tolerance solution of the original interval system.

## 1. Introduction

It is well-known that self-validating methods for linear systems demand a consideration of all errors in the coefficients of the system including round-off errors. A possible approach to treat linear systems with error control is to formulate the system as a linear interval system, that is to consider the system together with intervals corresponding to the errors in the respective coefficients of the system and to make use of interval arithmetic [11]. The latter is an efficient tool for the construction of self-validating methods due to the inclusion isotonicity of interval-arithmetic operations. However, the application of interval arithmetic to a verified solution of linear interval equations is intricate, because interval arithmetic does not permit a representation by coordinates so that interval operations, resp. interval algebraic problems, can not be generally reduced into real problems to be solved separately for each coordinate [6].

In order to treat interval algebraic equations involving multiplication of intervals we make use of a special "centered outward" multiplication of intervals. The centered outward interval multiplication produces outer enclosure for the familiar (set-theoretic) interval product. Throughout the paper we use midpoint-radius representation of intervals, which is a natural representation from both an algebraic and practical point of view [2], [3], [11]–[13].

**Interval addition and multiplication by scalars**. Denote by $\mathbb{R} = (\mathbb{R}, +, \cdot, \leq)$ the ordered field of reals and by $\mathbb{R}^n$ the $n$-dimensional real vector space. For $a = (a_1, a_2, ..., a_n)$, $b = (b_1, b_2, ..., b_n) \in \mathbb{R}^n$, the partial order "$\leq$" is given component-wise by $a \leq b \Longleftrightarrow a_i \leq b_i$ for all $i = 1, ..., n$. For $a', a'' \in \mathbb{R}^n$, $a'' \geq 0$, the set $A = \{x \in \mathbb{R}^n \mid a' - a'' \leq x \leq a' + a''\} = (a'; a'')$ is an ($n$-dimensional) *interval* (or *box*) in $\mathbb{R}^n$; $a'$ is the *midpoint (center)* of $A$ and $a'' \geq 0$ is the *radius* of $A$. The set of all $n$-dimensional intervals in $\mathbb{R}^n$ is denoted $I(\mathbb{R})^n$.

*Addition* of two intervals $A, B \in I(\mathbb{R})^n$ is defined by $A + B = \{c \mid c = a + b,\ a \in A,\ b \in B\}$, here $a, b, c \in \mathbb{R}^n$. *Multiplication of an interval* $B \in I(\mathbb{R})^n$ *by a real scalar* $\alpha \in \mathbb{R}$ is defined by $\alpha * B = \{c \mid c = \alpha b,\ b \in B\}$, $b, c \in \mathbb{R}^n$. *Negation* "$-$" is $-A = (-1) * A$, $A \in I(\mathbb{R})^n$. We have $-(\gamma * A) = (-1) * (\gamma * A) = (-\gamma) * A = \gamma * (-A)$ for $\gamma \in \mathbb{R}$ and $A \in I(\mathbb{R})^n$. *Subtraction* is $A - B = A + (-B)$. An interval $A \in I(\mathbb{R})^n$ is *symmetric*, if $A = -A$, and *degenerate*, if $A - A = 0$. In midpoint-radius form we have

$$
\begin{aligned}
(a'; a'') + (b'; b'') &= (a' + b'; a'' + b''), & (1)\\
\alpha * (b'; b'') &= (\alpha b'; |\alpha| b''), & (2)\\
-(a'; a'') &= (-a'; a''), & (3)\\
(a'; a'') - (b'; b'') &= (a' - b'; a'' + b''). & (4)
\end{aligned}
$$

Formulae (1)–(4) can be rewritten component-wise in the $n$-dimensional case, for instance (3) can be written as $-(a'_1; a''_1) = (-a'_1; a''_1)$, ... , $-(a'_n; a''_n) = (-a'_n; a''_n)$, or as:

$$-(a'_1, ..., a'_n; a''_1, ..., a''_n) = (-a'_1, ..., -a'_n; a''_1, ..., a''_n). \tag{5}$$

**Interval multiplication**. The set-theoretic multiplication of two intervals $A, B \in I(\mathbb{R})$ is:

$$A * B = \{\xi\eta \mid \xi \in A,\ \eta \in B\}. \tag{6}$$

Assume $A = (a'; a'')$, $B = (b'; b'') \in I(\mathbb{R})$, are such that $a' - a'' \geq 0$, $b' - b'' \geq 0$, then [11]:

$$\begin{aligned} A * B &= \{\xi \in \mathbb{R} \mid (a' - a'')(b' - b'') \leq \xi \leq (a' + a'')(b' + b'')\} \\ &= (a'b' + a''b'';\ b'a'' + a'b''). \end{aligned}$$

Denote by $I(\mathbb{R})^*$ the set of all intervals which do not contain zero as interior point: $I(\mathbb{R})^* = \{(a'; a'') \in I(\mathbb{R}) \mid a'' \leq |a'|\}$. Interval multiplication (6) in midpoint-radius form for the case $A, B \in I(\mathbb{R})^*$ can be written:

$$(a'; a'') * (b'; b'') = \begin{cases} (a'b' + a''b'';\ |b'|a'' + |a'|b''), & \text{if } a'b' \geq 0, \\ (a'b' - a''b'';\ |b'|a'' + |a'|b''), & \text{if } a'b' < 0. \end{cases} \tag{7}$$

Using $\sigma : \mathbb{R} \longrightarrow \Lambda = \{+, -\}$ defined by:

$$\sigma(\gamma) = \begin{cases} +, & \text{if } \gamma \geq 0, \\ -, & \text{if } \gamma < 0, \end{cases} \tag{8}$$

and the notation $\sigma a = \{a, \sigma = +;\ -a, \sigma = -\}$, formula (7) obtains the form:

$$(a'; a'') * (b'; b'') = (a'b' + \sigma(a'b')a''b'';\ |b'|a'' + |a'|b''), \tag{9}$$

whenever $a'' \leq |a'|$, $b'' \leq |b'|$. Interval multiplication by scalars (2): $\alpha * (b'; b'') = (\alpha b'; |\alpha| b'')$ is a special case of (7), resp. (9), considering $\alpha$ as a degenerate interval $\alpha = (\alpha; 0) \in I(\mathbb{R})$.

When (at least) one multiplier contains zero as an interior point (6) admits simple midpoint-radius representation. If $A = (a'; a'')$ is such that $a'' > |a'|$, then we have:

$$\begin{aligned} (a'; a'') * (b'; b'') &= (a'b' + \sigma(b')a'b'';\ |b'|a'' + a''b'') \\ &= (a'(b' + \sigma(b')b'');\ a''(|b'| + b'')) \\ &= (\sigma(b')a'(|b'| + b'');\ a''(|b'| + b'')), \end{aligned} \tag{10}$$

under the condition that either i) $b'' \leq |b'|$, or ii) $b'' > |b'|$, $|\varkappa(A)| \geq |\varkappa(B)|$, where $\varkappa(A) = a''/a'$, $a' \neq 0$, is the *relative extent (rex) function* introduced by Kulpa [2]. Using multiplication by scalars (10) can be written

$$(a'; a'') * (b'; b'') = (|b'| + b'') * (\sigma(b')a';\ a''). \tag{11}$$

In the special case of symmetric intervals we have $(0; a'') * (0; b'') = (0; a''b'')$.

**Inclusion of intervals** is expressed in midpoint-radius form by [5], [12]:

$$A \subseteq B \Longleftrightarrow |b' - a'| \leq b'' - a'', \quad A, B \in I(\mathbb{R}). \tag{12}$$

Inclusion isotonicity of addition: $A \subseteq B \Longleftrightarrow A + C \subseteq B + C$ is trivially demostrated in midpoint-radius coordinates. Indeed, in terms of (12) $A + C \subseteq B + C$ means $|b' + c' - (a' + c')| \leq b'' + c'' - (a'' + c'')$ which is equivalent to $|b' - a'| \leq b'' - a''$, that is $A \subseteq B$.

It is also easy to observe isotonicity of multiplication by real scalars: for $A, B \in I(\mathbb{R})$, $c \in \mathbb{R}$, $c \neq 0$,

$$A \subseteq B \Longleftrightarrow c * A \subseteq c * B. \tag{13}$$

Indeed, for $c \in \mathbb{R}$ fixed $c * A \subseteq c * B$ means in terms of (12): $|cb' - ca'| \leq |c|b'' - |c|a''$, which for $c \neq 0$ is equivalent to $|b' - a'| \leq b'' - a''$.

Using (6) and (13) it is easy to see that for $A, B, C \in I(\mathbb{R})$,

$$A \subseteq B \Longrightarrow C * A \subseteq C * B. \tag{14}$$

## 2. Inverse Inclusion Isotonicity of Interval Multiplication

To prove inverse inclusion isotonicity of interval multiplication, that is (14) in the inverse direction, we need the following

**Proposition 1** *If $A, B \in I(\mathbb{R})^*$, $C \in I(\mathbb{R})^* \setminus \{0\}$, $C * A \subseteq C * B$, then $\sigma(a') = \sigma(b')$.*

Proof. From the assumptions we have $\sigma(c'a') = \sigma(c'b')$, which implies $\sigma(a') = \sigma(b')$. □

**Theorem 1** *If $A, B, C \in I(\mathbb{R})$, $0 \notin C$, then*

$$C * A \subseteq C * B \Longrightarrow A \subseteq B. \tag{15}$$

Proof. By assumption $c'' < |c'|$. 1) We first consider the case $a'' \leq |a'|$, $b'' \leq |b'|$. Using Proposition 1 we see that the inclusion $C*A \subseteq C*B$ implies $\sigma(a') = \sigma(b') = \lambda$. In terms of (12) $C * A \subseteq C * B$ is equivalent to $|(C * B)' - (C * A)'| \leq (C * B)'' - (C * A)''$. Using (9), we obtain consecutively the equivalent inequalities

$$\begin{aligned} |b'c' + \lambda\sigma(c')b''c'' - a'c' - \lambda\sigma(c')a''c''| &\leq |b'|c'' + |c'|b'' - |a'|c'' - |c'|a'', \\ |(b' - a')c' + \lambda\sigma(c')(b'' - a'')c''| &\leq (|b'| - |a'|)c'' + |c'|(b'' - a''), \\ |\lambda(b' - a')|c'| + (b'' - a'')c''| &\leq \lambda(b' - a')c'' + |c'|(b'' - a''). \end{aligned}$$

Equivalently, for $\varepsilon \in \{+,-\}$, we have:

$$\begin{aligned}
\varepsilon(\lambda(b'-a')|c'| + (b''-a'')c'') &\leq \lambda(b'-a')c'' + |c'|(b''-a''),\\
\varepsilon\lambda(b'-a')|c'| - \lambda(b'-a')c'' &\leq |c'|(b''-a'') - \varepsilon(b''-a'')c'',\\
\varepsilon\lambda(b'-a')(|c'| - \varepsilon c'') &\leq (b''-a'')(|c'| - \varepsilon c'').
\end{aligned}$$

Using $|c'| - \varepsilon c'' > 0$, we obtain $\varepsilon\lambda(b'-a') \leq b''-a''$, that is $|b'-a'| \leq b''-a''$, which, according to (12) is equivalent to $A \subseteq B$.

2) Consider next the case $0 \in A$, $0 \in B$, $a' \neq 0$, $b' \neq 0$. Assume $\varkappa(A) \geq \varkappa(B)$ (otherwise, exchange $A$ and $B$). Using (11) we have $(a';a'') * (c';c'') = (|c'| + c'') * (\sigma(c')a';\ a'')$, $(b';b'') * (c';c'') = (|c'| + c'') * (\sigma(c')b';\ b'')$. The inclusion $C * A \subseteq C * B$ means, in terms of (12), $|(C*B)' - (C*A)'| \leq (C*B)'' - (C*A)''$, i. e. $(|c'|+c'')|\sigma(c')b' - \sigma(c')a'| \leq (|c'| + c'')(b''-a'')$, which is equivalent to $|b'-a'| \leq b''-a''$, that is to $A \subseteq B$. The case $a' = 0$ and/or $b' = 0$ is treated similarly.

3) Consider finally the "mixed" case when one of the intervals $A, B$ includes 0, the other does not. The case $0 \in A$, $0 \notin B$, is not possible; indeed, then we have the relations $0 \in C * A$, $0 \notin C * B$, which contradict the assumption $C * A \subseteq C * B$. Let $0 \notin A$, $0 \in B$, then $|(C*B)' - (C*A)'| \leq (C*B)'' - (C*A)''$ becomes

$$\begin{aligned}
|(|c'| + c'')\sigma(c')b' - a'c' - \sigma(a'c')a''c''| &\leq (|c'| + c'')b'' - |a'|c'' - |c'|a'',\\
\varepsilon((|c'| + c'')\sigma(c')b' - a'c' - \sigma(a'c')a''c'') &\leq (|c'| + c'')b'' - |a'|c'' - |c'|a''.
\end{aligned}$$

If $\varepsilon = -\sigma(a'c')$ then the above is equivalent to $-\sigma(a')(|c'| + c'')(b' - a') \leq (|c'| + c'')(b''-a'')$, or

$$-\sigma(a')(b'-a') \leq b''-a''. \tag{16}$$

If $\varepsilon = \sigma(a'c')$ then we obtain:

$$\begin{aligned}
\sigma(a')(|c'| + c'')b' - |a'||c'| - a''c'' &\leq (|c'| + c'')b'' - |a'|c'' - |c'|a'',\\
\sigma(a')(|c'| + c'')b' - |a'||c'| + |a'|c'' &\leq (|c'| + c'')b'' - |c'|a'' + a''c''.
\end{aligned}$$

Adding to the last inequality the inequality $2c''a'' \leq 2c''|a'|$, i. e. $-2c''|a'| \leq -2c''a''$, we obtain

$$\sigma(a')(|c'| + c'')b' - |a'||c'| - c''|a'| \leq (|c'| + c'')b'' - |c'|a'' - c''a'',$$

which is equivalent to $\sigma(a')(|c'| + c'')(b'-a') \leq (|c'| + c'')(b''-a'')$, that is to $\sigma(a')(b'-a') \leq b''-a''$. The latter, combined with (16), can be summarized as $|b'-a'| \leq b''-a''$, that is $A \subseteq B$. □

The proof of (15) using midpoint-radius coordinates is based on equivalent inequalities, hence it can also serve as a proof of (14) for the case $0 \notin C$. The conditions of Theorem 1 can be relaxed by replacing the assumption $0 \notin C$ by the less restrictive assumption $C \in I(\mathbb{R})^* \setminus \{0\}$.

**Proposition 2** *Assume that* $A = (a'; a'')$, $B = (b'; b'') \in I(\mathbb{R})$, $k \in \mathbb{R}$. *Then the following two assertions are equivalent:*

*i)* $|b' - a'| \leq k(b'' - a'')$, $k \geq 1$;

*ii)* $A \subseteq B + (0; r)$, $r = (k-1)(b'' - a'') \geq 0$.

Proof. According to (12) the inclusion $A \subset B + (0; r) = (b'; b'' + r)$, $r = (k-1)(b'' - a'') \geq 0$, is equivalent to $|b' - a'| \leq b'' + r - a'' = b'' + (k-1)(b'' - a'') - a'' = k(b'' - a'')$. □

**Remark**. Proposition 2 is a generalization of (12) in the sense that the latter is obtained from Proposition 2 for $k = 1$, resp. $r = 0$.

Consider the algebraic solution of the equation $A * X = B$. Assume $A, B \in I(\mathbb{R})^*$, $A \neq 0$, then it follows that $X \in I(\mathbb{R})^*$ as well. Using midpoint-radius representation and (9) we obtain the following equations for the midpoint-radius coordinates of $X$:

$$a'x' + \sigma(a')\sigma(x')a''x'' = b', \tag{17}$$

$$a''|x'| + |a'|x'' = b''. \tag{18}$$

Note that (17)–(18) is a not a linear system for $x'$, $x''$; indeed, we have to know the sign of $x'$ in advance to consider this system as a linear one. It is easy to determine $\sigma(x')$ in the case of one equation, but this may be tedious in the case of many equations. Another complication is that equations (17)–(18) are coupled. Similarly, a system of $n$ linear interval equations leads to a system of $2n$ equations for the coordinates of the algebraic solution. This shows that the coordinate systems corresponding to problems involving interval multiplication (of nondegenerate intervals) may be difficult to solve. To simplify the solution of such problems a special "centered outward" multiplication of intervals can be introduced, which produces an outward approximation of the set-theoretic interval multiplication and leads to simpler coordinate problems.

## 3. Centered Outward Interval Multiplication

Consider the following operation in $I(\mathbb{R})$:

$$a \circ b = (a'; a'') \circ (b'; b'') = (a'b';\ |a'|b'' + |b'|a'' + a''b''), \tag{19}$$

to be called *centered outward multiplication* of (proper) intervals, briefly: *co-multiplication*. Note that co-multiplication by real scalars (degenerate intervals: $a'' = 0$ or/and $b'' = 0$) coincides with the set-theoretic interval multiplication by scalars, $\alpha \circ (b'; b'') = \alpha * (b'; b'')$. Using the absolute value of $a \in I(\mathbb{R})$, defined as $|a| = |a'| + a''$, we can write (19) as: $a \circ b = (a'b';\ |a|b'' + |b'|a'') = (a'b';\ |b|a'' + |a'|b'')$.

The co-multiplication (19) produces generally wider results than the set-theoretic multiplication (6), as the following proposition shows

**Proposition 3** *For* $(a'; a''), (b'; b'') \in I(\mathbb{R})$ *we have*

$$(a'; a'') * (b'; b'') \subseteq (a'; a'') \circ (b'; b''). \tag{20}$$

Proof. Assume first that $(a'; a''), (b'; b'') \in I(\mathbb{R})^*$, that is the intervals do not contain zero as interior point. To prove inclusion (20) we apply formulae (12), (9) and (19) to obtain the obvious relation $|a''b''| \leq a''b''$, showing that (20) is true. Assume now that the first interval contains zero as an interior point, $a'' > |a'|$. Applying formulae (12), (10) and (19) we obtain the true relation $|a'b''| \leq |a'|b''$. This proves (20). $\square$

**Example 1.**
$(2; 0.1) * (4; 0.04) = (8.004; 0.480) = [7.524;\ 8.484]$,
$(2; 0.1) \circ (4; 0.04) = (8.000; 0.484) = [7.516;\ 8.484]$.

**Example 2.**
$(100; 1) * (100; 1) = (10001; 200) = [9801; 10201]$,
$(100; 1) \circ (100; 1) = (10000; 201) = [9799; 10201]$.

**Proposition 4** *Let* $A = (a'; a'')$, $0 \notin A$, $B = (b'; b'')$, $|\varkappa(A)| \leq |\varkappa(B)|$. *Then* $A \circ Y = B$ *has a unique algebraic solution* $Y = (y'; y'')$, *given by*

$$\begin{aligned} y' &= b'/a', \\ y'' &= (b'' - a''|b'/a'|)/|a|. \end{aligned}$$

*If* $X$ *is a solution to* $A * X = B$, *then* $Y \subseteq X$.

Proof. The equation $A \circ Y = B$ can be written in midpoint-radius form as $(a'y'; |a|y'' + |y'|a'') = (b'; b'')$, hence:

$$\begin{aligned} a'y' &= b', \\ |a|y'' + |y'|a'' &= b''. \end{aligned}$$

Using $a' \neq 0$, we can solve the first equation and replace the solution $y'$ in the second one to obtain:

$$\begin{aligned} y' &= b'/a', \\ |a|y'' &= b'' - a''|b'/a'|. \end{aligned}$$

Using $|a| = |a'| + a'' \neq 0$, we can solve the second equation. Hence the solution of $A \circ Y = B$ is $y' = b'/a'$, $y'' = (b'' - a''|b'/a'|)/|a|$. It is easy to check that $y'' \geq 0$ and that $0 \notin Y$. To see the latter, we calculate: $y''/|y'| = (|a'/b'|b'' - a'')/(|a'| + a'') \leq (|a'/b'|b'' - a'')/|a'| = b''/|b'| - a''/|a'| \leq b''/|b'| = \varkappa(B) \leq t^{-1}$. Let $X$ be a solution to $A * X = B$. From $A * Y \subseteq A \circ Y = A * X$, using (15), we obtain $Y \subseteq X$. □

**Proposition 5** *For $A, B, C \in I(\mathbb{R})^*$, we have*

$$A \subseteq B \Longrightarrow A \circ C \subseteq B \circ C. \tag{21}$$

Proof. We have to prove that $A \subseteq B$, that is $|b' - a'| \leq b'' - a''$, implies $A \circ C \subseteq B \circ C$. According to (12) the latter is equivalent to the following inequalities:

$$\begin{aligned} |b'c' - a'c'| &\leq |b'|c'' + |c'|b'' + b''c'' - |a'|c'' - |c'|a'' - a''c'', \\ |(b' - a')c'| &\leq (|b'| - |a'|)c'' + (|c'| + c'')(b'' - a''), \\ \varepsilon(b' - a')|c'| &\leq \lambda(b' - a')c'' + (|c'| + c'')(b'' - a''), \\ \varepsilon(b' - a')(|c'| - \lambda\varepsilon c'') &\leq (|c'| + c'')(b'' - a''), \end{aligned} \tag{22}$$

where $\lambda = \sigma(a') = \sigma(b')$, $\varepsilon \in \{+, -\}$. By assumption $c'' \leq |c'|$, hence $0 \leq |c'| - \lambda\varepsilon c''$. This and $|b' - a'| \leq b'' - a''$ imply (22), which is equivalent to $A \circ C \subseteq B \circ C$. This proves (21). □

It is worth to note that Theorem 1 is not true if the set-theoretic interval multiplication "$*$" is replaced by the interval co-multiplication "$\circ$": $A \circ C \subseteq B \circ C$ does not imply $A \subseteq B$. This can be demonstrated by the following example.

**Example 3**. Assume $A = (1; 1) = [0, 2]$, $B = C = (3; 2) = [1, 5]$. Then, $A \circ C = (3; 7) = [-4, 10]$, $B \circ C = (9; 16) = [-7, 25]$. We have $A \circ C \subseteq B \circ C$, but $A \subseteq B$ does not hold.

The next proposition shows that if we "slightly" expand $B$, then an inclusion holds.

**Proposition 6** *Assume that $A = (a'; a'')$, $B = (b'; b'')$, $C = (c'; c'') \in I(\mathbb{R})$, $0 \notin C$. Then*

$$C \circ A \subseteq C \circ B \Longrightarrow A \subseteq B + (0; r),$$

*wherein $r = (k - 1)(b'' - a'') \geq 0$, $k \geq (|c'| + c'')/(|c'| - c'') \geq 1$.*

Proof. According to (12) the inclusion $C \circ A \subseteq C \circ B$ is equivalent to

$$\begin{aligned} |b'c' - a'c'| &\leq |b'|c'' + |c'|b'' + b''c'' - |a'|c'' - |c'|a'' - a''c'', \\ |c'(b' - a')| &\leq (|b'| - |a'|)c'' + |c'|(b'' - a'') + (b'' - a'')c'', \\ |c'||b' - a'| &\leq |b' - a'|c'' + (|c'| + c'')(b'' - a''), \\ (|c'| - c'')|b' - a'| &\leq (|c'| + c'')(b'' - a''). \end{aligned}$$

Thus we obtain $|b' - a'| \leq k(b'' - a'')$, with $k + (|c'| + c'')/(|c'| - c'') \geq 1$. Using Proposition 2 we see that $A \subseteq B + (0; r)$, where $r = (k-1)(b'' - a'') \geq 0$, $k \geq (|c'| + c'')/(|c'| - c'') \geq 1$. □

The co-multiplication (19) has been proposed in [12]; independently it has been introduced and practically implemented in [11] and has been studied in [4] with respect to distributivity. As noted in [4] the operation (19) is a special case of the complex disc multiplication introduced in [1]. A software implementation of the co-multiplication is reported in [9]; it has been shown that *overestimation* of co-multiplication satisfies $(A * B)''/(A \circ B)'' \leq 1.5$ and is globally limited [10].

## 4. Linear Interval Systems

In the previous sections we use upper case letters to denote intervals; in the sequel we use lower case letters for intervals and interval vectors and upper case letters for interval matrices. Assume that $C = (c_{ij}) \in I(\mathbb{R})^{n \times n}$ is a square matrix of intervals and $a = (a_j) \in I(\mathbb{R})^n$ is an interval vector. Denote $C * a = \{\sum_{j=1}^n c_{ij} * a_j\}_{i=1}^n \in I(\mathbb{R})^n$, resp. $C \circ a = \{\sum_{j=1}^n c_{ij} \circ a_j\}_{i=1}^n \in I(\mathbb{R})^n$. Using (20) we see that

$$C * a \subseteq C \circ a, \quad C \in I(\mathbb{R})^{n \times n}, \quad a \in I(\mathbb{R})^n.$$

We write $C = (C'; C'')$, with $C' = (c'_{ij}) \in \mathbb{R}^{n \times n}$, $C'' = (c''_{ij}) \in \mathbb{R}^{n \times n}$, $c''_{ij} \geq 0$, similarly $a = (a'; a'')$, $a' \in \mathbb{R}^n$, $a'' \in \mathbb{R}^n$, $a'' \geq 0$.

The *absolute value of an interval* $c \in I(\mathbb{R})$ is $|c| = |c'| + c''$. Correspondingly, we have for the interval matrix $C \in I(\mathbb{R})^{n \times n}$, $|C| = (|c_{ij}|) = |C'| + C''$, and for the interval vector $c \in I(\mathbb{R})^n$, $|c| = (|c_i|) = |c'| + c''$.

Recall that $C \in I(\mathbb{R})^{n \times n}$ is a *regular* interval matrix, if every real matrix $\tilde{C} \in \mathbb{R}^{n \times n}$, $\tilde{C} \in C$, is nonsingular [5]; $C \in I(\mathbb{R})^{n \times n}$ is a *nonnegative* interval matrix, if every real matrix $\tilde{C} \in \mathbb{R}^{n \times n}$, $\tilde{C} \in C$, is nonnegative; $C \in I(\mathbb{R})^{n \times n}$ is an *inverse nonnegative* interval matrix, if every real matrix $\tilde{C} \in \mathbb{R}^{n \times n}$, $\tilde{C} \in C$, is inverse nonnegative (that is $\tilde{C}^{-1}$ is nonnegative).

**Algebraic solutions.** Consider a linear interval $(n \times n)$-system of the form

$$A * x = b, \tag{23}$$

where $A = (a_{ij}) \in I(\mathbb{R})^{n\times n}$ is a regular matrix of intervals and $b \in I(\mathbb{R})^n$ is an interval vector. A vector $x \in I(\mathbb{R})^n$ satisfying (23) is called *algebraic solution* to (23), see, e. g. [7]–[8].

As mentioned at the end of Section 2, the interval system (23) leads to a system of $2n$ algebraic equations for the coordinates of $x$, which is not linear. Using the method for one equation, see Proposition 5, consider the associated system

$$A \circ y = b, \tag{24}$$

obtained from (23) by replacing all interval multiplications "$*$" by co-multiplications "$\circ$" defined by (19). Denoting $A = (A'; A'')$, $b = (b'; b'')$, $y = (y'; y'')$, we have in matrix notation

$$A \circ y = (A'y'; |A'|y'' + A''|y'| + A''y'') = (A'y'; |A|y'' + A''|y'|), \tag{25}$$

using $|A| = |A'| + A''$. From (24), (25) we obtain the following real system of $2n$ equations for the midpoint-radius coordinates of $y$:

$$A'y' = b', \tag{26}$$
$$|A|y'' = b'' - A''|y'|. \tag{27}$$

System (26)–(27) can be solved in two steps: first solve system (26) using that $A'$ is nonsingular ($A$ is assumed regular) to obtain a solution $y' = (A')^{-1}b'$, then substitute $y'$ in (27) to obtain the system

$$|A|y'' = b'' - A''|(A')^{-1}b'|. \tag{28}$$

To solve the latter system, we must assume that $|A| = |A'| + A''$ is nonsingular, then

$$y'' = |A|^{-1}(b'' - A''|(A')^{-1}b'|).$$

If $y'' \geq 0$, then the right-hand side of (28) is necessarily nonnegative, $d = b'' - A''|(A')^{-1}b'| \geq 0$. The latter condition can be written in the form $b'' \geq A''|(A')^{-1}b'|$, showing that the right-hand side vector $b$ is "sufficiently wide". The situation is analogous to the one in the one-dimensional case, see Proposition 5, where we assume $|\varkappa(B)| \geq |\varkappa(A)|$. If $y'' \geq 0$, that is $|A|^{-1}d \geq 0$, then $y = (y'; y'')$ is a unique (algebraic) solution to (24). We proved the following

**Proposition 7** *Assume that the interval matrix $A \in I(\mathbb{R})^{n\times n}$ is regular and that the matrix $|A|$ is nonsingular. Assume also $|A|^{-1}d \geq 0$, where $d = b'' - A''|(A')^{-1}b'|$. Then there exists an interval vector $y \in I(\mathbb{R})^n$, satisfying problem (24).*

**Tolerance solution**. The *tolerance solution* to (23) is the set: $\{\eta \in \mathbb{R}^n \mid A * \eta \subseteq b\}$. The (algebraic) solution to problem (24) (whenever existing) is connected with the tolerance solution to system (23). To demonstrate this, assume that $y \in I(\mathbb{R})^n$ is a solution to (24). From $A * y \subseteq A \circ y = b$ we see that $y$ satisfies the inclusion $A * y \subseteq b$. As any real $\eta \in y$ satisfies $A * \eta \subseteq b$, we see that the interval vector $y$ is included in the tolerance solution $\{\eta \in \mathbb{R}^n \mid A * \eta \subseteq b\}$ to (23). On the base of the above arguments we can formulate:

**Theorem 2** *Assume that the interval matrix $A \in I(\mathbb{R})^{n\times n}$ is regular and $|A|$ is nonsingular. Assume also that $|A|^{-1}(b'' - A''|(A')^{-1}b'|) \geq 0$. Then the interval vector $y \in I(\mathbb{R})^n$, satisfying (according to Proposition 7) problem (24), is an inner inclusion of the tolerance solution to (23).*

**The special case of a real matrix**. Consider the case when the matrix $A$ is a real matrix (all entries in $A$ are degenerate intervals). Then the matrix $A''$ is the null matrix and for $x = (x'; x'') \in I(\mathbb{R})^n$ we have $A * x = A \circ x = (Ax'; |A|x'')$. Systems (26)–(27) obtain the simple form

$$Ax' = b', \tag{29}$$

$$|A|\, x'' = b''. \tag{30}$$

Assuming that the real matrices $A$ and $|A|$ are nonsingular, we obtain for the solution of (29)–(30): $x' = A^{-1}b'$, $x'' = |A|^{-1}b''$. We must assume $|A|^{-1}b'' \geq 0$, so that $x'' \geq 0$. We thus obtain the following corollary:

**Corollary**. Given system (23), such that $A \in \mathbb{R}^{n\times n}$, $b = (b', b'') \in I(\mathbb{R})^n$, assume that the real matrices $A$ and $|A|$ are nonsingular and $|A|^{-1}b'' \geq 0$. Then there exists an unique interval solution $x$ to (23).

**Concluding remarks**. Throughout the work we systematically use the properties of the midpoint-radius presentation of intervals, interval vectors and interval matrices. We prove inverse isotonicity of (set-theoretic) interval multiplication and show that co-multiplication is not inverse inclusion isotone.

We demonstrate that a linear system involving intervals and interval co-multiplication can be reduced to real linear systems for the midpoint-radius coordinates of the intervals. Under certain assumptions these real coordinate systems produce inner estimates for the tolerance solutions of the original linear interval system. As a special case we show that a linear system with interval right-hand side and an exact real matrix can be reduced to two real linear systems for the midpoint-radius coordinates of the intervals.

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## References

[1] Henrici, P., *Circular Arithmetic and the Determination of Polynomial Zeroes*, Springer Lecture Notes in Mathematics 228 (1971), 86–92.

[2] Kulpa, Z., *Diagrammatic representation for interval arithmetic*, Linear Algebra and its Applications 324 (2001), 55–80.

[3] Markov, S., K. Okumura, *The Contribution of T. Sunaga to Interval Analysis and Reliable Computing*, In: T. Csendes (ed.) Developments in Reliable Computing, Kluwer, 1999, 167–188.

[4] Neumaier, A., *A Distributive Interval Arithmetic*, Freiburger Intervall–Berichte 82/10, Inst. f. Angew. Math., U. Freiburg i. Br. (1982), 31–38.

[5] Neumaier, A., *Interval Methods for Systems of Equations*, Cambridge University Press, 1990.

[6] Ratschek, H., *Representation of Interval Operations by Coordinates*, Computing 24 (1980), 93–96.

[7] Rohn, J., *Systems of Linear Interval equations.* Linear Algebra and its Applications 126 (1989), 39–78.

[8] Rohn, J., *Interval Solutions of Linear Interval Equations*, Applicace Matematiky 35 (1990), 3, 220–224.

[9] Rump, S. M., *INTLAB — INTerval LABoratory*, In: T. Csendes (ed.) Developments in Reliable Computing, Kluwer, 1999, 77–104.

[10] Rump, S. M., *Fast and Parallel Interval Arithmetic*, BIT 39 (1999), 3, 534–554.

[11] Sunaga, T., *Theory of an Interval Algebra and its Application to Numerical Analysis*, RAAG Memoirs 2 (1958), Misc. II, 547–564.

[12] Warmus, M., *Calculus of Approximations*, Bull. Acad. Polon. Sci., Cl. III 4 (1956), 253–259.

[13] Warmus, M., *Approximations and Inequalities in the Calculus of Approximations. Classification of Approximate numbers*, Bull. Acad. Polon. Sci., Ser. math. astr. et phys., 9 (1961), 241–245.