# On the solution to numerical problems using stochastic arithmetic


René Alt, Jean-Luc Lamotte
CNRS, UMR 7606, LIP6
University Pierre et Marie Curie
4 place Jussieu
75252 Paris cedex 05, France
Rene.Alt@lip6.fr
Jean-Luc.Lamotte@lip6.fr

Svetoslav Markov
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
"G. Bonchev" st.
bl. 8, 1113 Sofia, Bulgaria
smarkov@bio.bas.bg



## Abstract

*We investigate some algebraic properties of the system of stochastic numbers with the arithmetic operations addition and multiplication by scalars and the relation inclusion and point out certain practically important consequences from these properties. Our idea is to start from a minimal set of empirically known properties and to study these properties by an axiomatic approach. Based on this approach we develop an algebraic theory of stochastic numbers. A numerical example based on the Lagrange polynomial demonstrates the consistency between the CESTAC method and the presented theory of stochastic numbers.*


## 1. Introduction

The CESTAC method is a widely used statistical Monte-Carlo type method to compute the accuracy of the solution of real life numerical problems implemented on a computer. In this method an imprecise numbers is represented as an $N$-tuple of random values with Gaussian distribution so that the mean value and standard deviation of this $N$-tuple provide respectively approximations of the exact unknown value and of the error. Such an $N$-tuple which is a sampling of a Gaussian random variable is named in this context a *discrete stochastic number*. In practice the CESTAC method has been implemented in a software called CADNA in which the $N$ samples of the stochastic numbers are randomly rounded up or down so as to take into account the round-off errors, see [3], [8], [9], [12], with the same idea that directed rounding is used for implementing interval arithmetic [5].

In order to provide a good algebraic understanding of the performance of the CESTAC method discrete stochastic numbers have been modeled as Gaussian random variables with a known mean value and a known standard deviation named here *stochastic numbers*. Thus stochastic numbers are idealizations of discrete stochastic numbers. To get an idea of the operations between stochastic numbers it should be mentioned that we are concerned with imprecise data in which the unknown errors are relatively important (say, of order $10^{-2}$–$10^{-3}$) whereas the arithmetic operations are performed using double precision arithmetic. In this case it can be easily checked that addition and multiplication by reals in the CESTAC method satisfy certain algebraic properties, e. g. stochastic numbers form a commutative and cancellative monoid with respect to addition. Some fundamental properties of stochastic numbers are considered in [4], [10].

This work is part of a more general one, which aims,to study the abstract algebraic structures induced by the operations on stochastic numbers and to compare the relevant theory with the performance of the CESTAC method [1], [2], [6], [7]. In the paper we restrict ourselves to the arithmetic operations addition, negation and multiplication by scalars and the relation inclusion.

### 1.1. Stochastic Arithmetic (*CSA*): basic properties

Denote by $S$ the set of stochastic numbers and by $S^n$ the set of all $n$-tuples of stochastic numbers. The operations addition and multiplication by $-1$ (negation) are well defined in $S$, resp. $S^n$. We next recall some basic properties of these operations and derive some logical consequences of these properties, which will allow us to better understand the nature of stochastic numbers. The system $(S^n, +)$ is a commutative monoid (semigroup with null) with cancellation law. The operator negation is an automorphism $\neg : S^n \to S^n$, that is: $\neg(A + B) = \neg A + (\neg B)$, and involution: $\neg(\neg A) = A$. These properties can be checked experimentally, say, by a CESTAC-like method, see e. g.

[8], [9], [10]. So, we next consider these properties as axiomatically given, and we want to derive some simple consequences.

We first note that $S^n$ is not a group with respect to addition; however it can be easily embedded in a group. The standard algebraic construction that converts any abelian monoid with cancellation law into a group will be further referred as embedding construction. Recall that this approach is used to pass from the monoid of nonnegative reals $(\mathbb{R}^+, +)$ to the set of reals $(\mathbb{R}, +)$. Thus, it is natural instead of the original system $(S^n, +)$ to consider the extended system $(\mathbb{S}^n, +)$ obtained by the embedding construction. We next briefly recall this construction.

### 1.2. Algebraic completion of the monoid of stochastic numbers

Every abelian monoid $(M, +)$ with cancellation law induces an abelian group $(\mathbb{M}, +)$, where $\mathbb{M} = M^2/\sim$ is the *difference (quotient) set* of $M$ consisting of all pairs $(A, B)$ factorized by the congruence relation $\sim$: $(A, B) \sim (C, D)$ iff $A+D = B+C$, for $A, B, C, D \in M$. Addition in $\mathbb{M}$ is defined by $(A, B) + (C, D) = (A + C, B + D)$. The neutral (null) element of $\mathbb{M}$ is the class $(Z, Z)$, $Z \in M$. Due to the existence of null element in $M$, we have $(Z, Z) \sim (0, 0)$. The opposite element to $(A, B) \in \mathbb{M}$ is $\text{opp}(A, B) = (B, A)$. The mapping $\varphi : M \longrightarrow \mathbb{M}$ defined for $A \in M$ by $\varphi(A) = (A, 0) \in \mathbb{M}$ is an *embedding* of monoids. We *embed* $M$ in $\mathbb{M}$ by identifying $A \in M$ with the equivalence class $(A, 0) \sim (A + X, X)$, $X \in M$; all elements of $\mathbb{M}$ admitting the form $(A, 0)$ are called *proper* and the remaining (new) elements are called *improper*. The set of all proper elements of $\mathbb{M}$ is $\varphi(M) = \{(A, 0) \mid A \in M\} \cong M$.

Using the embedding construction described above the system $(S^n, +)$ is embedded into a group $(\mathbb{S}^n, +)$ in a unique way. We define multiplication by $-1$, called *negation*, in the group $\mathbb{S}^n$, by means of: $\neg(A, B) = (\neg A, \neg B)$, $A, B \in S^n$. In what follows we shall use lower case roman letters to denote the elements of $\mathbb{S}^n$, writing e. g. $a = (A_1, A_2), A_1, A_2 \in \mathbb{S}^n$. Thus negation will be denoted as $\neg a$; $a \neg b$ means $a + (\neg b)$. Clearly the properties of negation in $S$ hold also in $\mathbb{S}$, that is: $\neg(a + b) = \neg a \neg b$ and $\neg(\neg a) = a$. From $\text{opp}(a) + a = 0$ we obtain $\neg\text{opp}(a) \neg a = 0$, that is $\neg\text{opp}(a) = \text{opp}(\neg a)$. The element $\neg\text{opp}(a) = \text{opp}(\neg a)$ is further denoted by $a_-$ and the corresponding operator is called *dualization* or *conjugation*. We say that $a_-$ is the conjugate (or dual) of $a$. In the sequel we shall express the opposite element symbolically as: $\text{opp}(a) = \neg a_-$, minding that $a + (\neg a_-) = 0$ (to be briefly written $a \neg a_- = 0$).

In Section 2 we investigate the system $(\mathbb{S}^n, +, \neg)$ obtained by algebraic completion by means of a novel approach. Namely, starting from a minimal set of basic algebraic properties, we naturally arrive to the necessity of studying separately the spaces of mean values (which is a vector space) and of standard deviations (which is an s-space). In Section 3 we consider the system $(\mathbb{S}^n, +, \mathbb{R}, *)$. There we also discuss an algebraically natural approach to define an order relation inclusion for stochastic numbers arriving thus to the system $(\mathbb{S}^n, +, \mathbb{R}, *, \subseteq)$. In Section 4 we give some numerical examples aiming to compare the CSA theory with the performance of the CESTAC method. Our numerical experiments with Lagrange interpolation demonstrate that the model of stochastic numbers is, at least in the case of the operationsin consideration, in perfect agreement with the results obtained with the CESTAC method.

## 2. Decomposing the group of stochastic numbers

We shall now concentrate on the algebraic properties of the system $(\mathbb{S}^n, +, \neg)$. For a better understanding the following reminder about expressions involving the dual operator will be useful: i) $a + \text{opp}(a) = 0$ is equivalent to $a_- \neg a = 0$ or $a \neg a_- = 0$; ii) $(a + b)_- = a_- + b_-$, iii) $y \neg y = 0$ is equivalent to $y = y_-$; iv) $\neg z = z$ is equivalent to $z + z_- = 0$. An element $y$ with property iii) is called *linear* or *distributive*; an element $z$ with property iv) is called *centred* or *0-symmetric*.

Denote $nx = x + x + ... + x$ ($n$ times). We recall that a divisible (additive) group is such that every equation of the form $nx = a$ has a solution $x$ for any $a \in G$; the solution will be further denoted $(1/n)a$. An (additive) group is torsion-free if $na = 0$ implies $a = 0$ for any $a \in G$.

*Remark.* Clearly, the group of stochastic numbers is divisible and torsion-free (the monoid possesses the same properties). For our purposes it will be sufficient that the following two properties hold in $(G, +)$: i) $x + x = a \Longrightarrow x = (1/2)a$, and ii) $x + x = 0 \Longrightarrow x = 0$. Note that the latter property is equivalent to $x + x = y + y \Longrightarrow x = y$ (indeed, $x + x = y + y \Longrightarrow (x \neg y_-) + (x \neg y_-) = 0 \Longrightarrow (x \neg y_-) = 0 \Longrightarrow x = y$).

Let $(G, +)$ be an additive abelian divisible torsion-free group. In addition we shall assume that $G$ posseses an involutary automorphism $\neg : G \to G$, such that: C1. $\neg(a + b) = \neg a \neg b$; C2. $\neg(\neg a) = a$.

*Remark.* In particular, the operator "$\neg$" may coincide with opposite or identity. Conditions C1–C2 imply $\neg 0 = 0$ (to see this substitute $b = \neg a_-$ in C1).

**Theorem** (Decomposition theorem). $G$ is an additive divisible torsion-free abelian group with an involutary automorphism "$\neg$". For every $x \in G$ there exist unique $y, z \in G$, such that: i) $x = y + z$; ii) $y \neg y = 0$; iii) $\neg z = z$; iv) $y = z \Longrightarrow y = z = 0$.

*Proof.* Let us consider equations i)–iii) as a system of equations for $x, y, z$ and let us solve this system with respect to $y$ and $z$. To this end we shall repeatedly use some properties of the dual operator, such that opp$(a) = \neg a_-$, equation $y \neg y = 0$ is equivalent to $y = y_-$ and $\neg z = z$ is equivalent to $z + z_- = 0$.

Write $x = y + z$ in the form: $x_- = y_- + z_-$; replacing $y_-$ by $y$ and $z$ by $\neg z$ we obtain: $x_- = y \neg z_-$. Adding the latter equation to $x = y + z$ we obtain

$$x + x_- = y + y. \tag{1}$$

Similarly, write $x = y + z$ in the form :$\neg x = \neg y + \neg z$; replacing $y_-$ by $y$ and $z$ by $\neg z$ we obtain: $\neg x = \neg y_- + z$. Adding the latter equation to $x = y + z$ we obtain

$$x \neg x = z + z. \tag{2}$$

Summing up equations (1) and (2) we obtain $x + x = y + y + z + z$, which implies $x = y + z$ (using that $x + x = 0 \Longrightarrow x = 0$).

Assume now that a $x \in G$ is given. Chose $y$ and $z$ to satisfy resp. (1) and (2). Such elements exist due to the divisibility assumption; denote them $y = (1/2)(x + x_-)$ and $z = (1/2)(x \neg x)$. Clearly, $y$ is linear, whereas $z$ is centred. As we have $x = y + z$, it follows that any $x \in G$ is decomposable into a sum of a linear and a centred element. To show that the sum is direct, it remains to prove that $y = z \Longrightarrow y = z = 0$. Assume $y = z$. Then $y + y = z + z$, or $x + x_- = x \neg x$. The latter implies $x_- = \neg x$, or $x = \neg x_- = \text{opp}(x)$, or $x + x = 0$, or $x = 0$ (again due to the assumption $x + x = 0 \Longrightarrow x = 0$). □

**Comments**. The Decomposition theorem states that any $x \in G$ can be written in the form:

$$x = y + z = (1/2)(x + x_-) + (1/2)(x \neg x). \tag{3}$$

The element $y = (1/2)(x + x_-)$ satisfies $y = y_-$, equivalently $y \neg y = 0$; thus $y$ is a linear (distributive) element.

Alternatively, the element $z = (1/2)(x \neg x)$ satisfies $z = \neg z$, equivalently $z + z_- = 0$; therefore $z$ is centred (0-symmetric).

The subset of all linear elements of $G$ is denoted $G' = \{y \in G \mid y \neg y = 0\}$ and the subset of all centred elements of $G$ is denoted $G'' = \{z \in G \mid z = \neg z\}$.

If negation coincides with opposite in $G$, then $x \neg x = 0$ for all $x \in G$ and, from (3): $x = (1/2)(x + x_-)$ for all $x \in G$. In this case the subset $G''$ is empty and $G$ consists only of linear elements. Alternatively, if negation coincides with identity in $G$, then $x + x_- = 0$ for all $x \in G$ and, from (3): $x = (1/2)(x \neg x)$ for all $x \in G$. In this case the subset $G'$ is empty and $G$ consists only of centred elements.

**Corollary**. Let $G$ be an additive divisible torsion-free abelian group with an involutary automorphism "$\neg$". Then $G$ is a direct sum of $G' = \{y \in G \mid y \neg y = 0\}$ and $G'' = \{z \in G \mid z = \neg z\}$, symbolically $G = G' \bigoplus G''$.

*Remark.* Note that while the group $G = (G, +, \neg)$ possesses an operator (negation) in addition to opposite, the subgroups $G'$, $G''$ do not possess additional operator (as negation coincides with opposite in $G'$ and with identity in $G''$). Therefore we do not need to write down the operator negation in the groups: $G = G' \bigoplus G''$ can be written as $(G, +, \neg) = (G', +) \bigoplus (G'', +)$.

## 2.1. Practical consequences

The Decomposition theorem implies that the algebraically natural presentation of stochastic numbers is as a sum of two components — a linear component and a centred component. The linear component, by definition, is such that negation of this element coincides with opposite, and the centred component is such that negation coincides with identity. The latter holds for elements of $\mathbb{S}$, but it also holds for elements of $S$ (that is proper stochastic numbers). Indeed, by the embedding construction, proper stochastic numbers as elements of $\mathbb{S}$ are pairs of the form $(A, 0)$, wherein $A \in S$. Assume first that $(A, 0)$ is linear, that is $(A, 0) \neg (A, 0) = 0$; this implies $A \neg A = 0$, that is negation of $A$ coincides with opposite. Such property have, e. g., real numbers (real vectors) as elements of a linear space. Assume now that $(A, 0)$ is centred, that is $(A, 0) = \neg (A, 0)$; this means $A = \neg A$, that is negation of $A$ coincides with identity.

According to the Corollary the group of stochastic numbers $\mathbb{S}$ decomposes as a direct sum of two subgroups $\mathbb{S} = \mathbb{S}' \bigoplus \mathbb{S}''$. Thus a stochastic number $a \in \mathbb{S}$ can be written in the form $a = (a'; a'')$. As well-known, the first component $a'$ is interpreted as mean value, and the second component $a''$ is the standard deviation. A stochastic number of the form $(a'; 0)$ has zero standard deviation and represents a (pure) mean value, whereas a stochastic number of the form $(0; a'')$ has zero mean value and represents a (pure) standard deviation. Using addition in $\mathbb{S}$, as defined in a direct sum by means of $(a'; a'') + (b'; b'') = (a' + b'; a'' + b'')$, we have the presentation $a = (a'; a'') = (a'; 0) + (0; a'')$. For negation we have $\neg a = \neg(a'; a'') = (\neg a'; \neg a'') = (-a'; a'')$, minding that $\neg a' = -a'$ and $\neg a'' = a''$.

In the case of stochastic vectors we have: $a = (a'; a'') \in \mathbb{S}^n$ with $a' \in \mathbb{R}^n$, $a'' \in \mathbb{R}^n$. We have $a'' \geq 0$ for $a$ proper; $a''$ has at least one negative component for improper $a$.

The two systems $(\mathbb{S}', +)$, $(\mathbb{S}'', +)$ composing $(\mathbb{S}, +, \neg)$ are of distinct algebraic nature, hence it is correct to use different notations for the operation addition. The system of mean values is identified with the additive group of reals $(\mathbb{R}, +)$ where negation coincides with opposite: $\neg \delta = -\delta$ for $\delta \in \mathbb{R}$. Therefore we shall use the usual sign for addition in the system $(\mathbb{S}', +)$, but we use a distinct sign for addition in the system $(\mathbb{S}'', +)$, namely $(\mathbb{S}'', \oplus)$.

In order to characterize the system $(\mathbb{S}'', \oplus)$ define for $\alpha \in \mathbb{R}$ the *sign function* $\sigma$ by: $\sigma(\alpha) = \{+, \text{ if } \alpha \geq 0; \ -, \text{ if } \alpha < 0\}$. As shown in [6], [7], the system $(\mathbb{S}'', \oplus)$ is identified with the group system $(\mathbb{R}, \oplus)$, where for $\alpha, \beta \in \mathbb{R}$ addition is defined by:

$$\alpha \oplus \beta = \sigma(\alpha + \beta)\sqrt{|\sigma(\alpha)\alpha^2 + \sigma(\beta)\beta^2|}, \quad (4)$$

and negation is identity: $\neg\delta = \delta$ for $\delta \in \mathbb{R}$.

Define the *symmetric square and square root functions* by: $\alpha^{\widehat{2}} = \sigma(\alpha)\alpha^2$, $\alpha^{\widehat{1/2}} = \sigma(\alpha)|\alpha|^{1/2}$, $\alpha \in \mathbb{R}$. If $\alpha \geq 0$ then $\alpha^{\widehat{2}}$ coincides with $\alpha^2$, similarly $\alpha^{\widehat{1/2}}$ and $\alpha^{1/2}$ are identical; however if $\alpha < 0$ then $\alpha^{\widehat{2}} = -\alpha^2$, $\alpha^{\widehat{1/2}} = -\alpha^{1/2}$. Using the symmetric square and square root functions (4) can be written as

$$\alpha \oplus \beta = (\alpha^{\widehat{2}} + \beta^{\widehat{2}})^{\widehat{1/2}}. \quad (5)$$

For two stochastic numbers $(m_1; s_1),\ (m_2; s_2) \in \mathbb{S}$, we have

$$\begin{aligned} (m_1; s_1) + (m_2; s_2) &= (m_1 + m_2; (s_1^{\widehat{2}} + s_2^{\widehat{2}})^{\widehat{1/2}}), \quad (6) \\ \neg(m_1; s_1) &= (-m_1;\ s_1). \quad (7) \end{aligned}$$

For the special case of two proper stochastic numbers $(m_1; s_1),\ (m_2; s_2)$, $s_1, s_2 \geq 0$, formula (6) becomes

$$(m_1; s_1) + (m_2; s_2) = (m_1 + m_2;\ \sqrt{s_1^2 + s_2^2}). \quad (8)$$

This is the familiar formula for the addition of two independent random variables with normal distribution. The advantage of formula (6) is that it can be applied in the general case without assuming nonnegativity of standard deviations.

# 3. The space $(\mathbb{S}^n, +, \mathbb{R}, *, \subseteq)$

The Decomposition theorem states that a group with negation is a direct sum of two spaces under mild assumptions for the group (divisibility, torsion-freedom) and the negation operator (automorphism, involution). As the groups $\mathbb{S}$, resp. $\mathbb{S}^n$, satisfy these assumptions (as known from empirical evidence), we can state that $\mathbb{S} = \mathbb{S}' \bigoplus \mathbb{S}''$, resp. $\mathbb{S}^n = (\mathbb{S}')^n \bigoplus (\mathbb{S}'')^n$, where $\mathbb{S}'$ is the space of mean values and $\mathbb{S}''$ is the space of standard deviations.

## 3.1. Multiplication by scalars

Multiplication by scalars "$*$" is defined for $x = (m; s) \in \mathbb{S}$ by

$$\gamma * x = \gamma * (m; s) \stackrel{def}{=} (\gamma m;\ |\gamma| s),\ \gamma \in \mathbb{R}. \quad (9)$$

Clearly, the system $(\mathbb{S}', +, \mathbb{R}, *)$ is a linear space and will be written also as $(\mathbb{S}', +, \mathbb{R}, \cdot)$. As we know the system $(\mathbb{S}'', \oplus, \mathbb{R}, *)$, resp. $((\mathbb{S}'')^n, \oplus, \mathbb{R}, *)$, with operation addition defined by (4) and multiplication by scalars defined from (9) by

$$\gamma * s = |\gamma| s,\ \gamma \in \mathbb{R}, \quad (10)$$

is an *s-space* in the sense of the following definition [6]:

**Definition**. The system $(\mathcal{G}, \oplus, \mathbb{R}_D, *)$ is called an s-space if $(\mathcal{G}, \oplus)$ is an abelian group, such that for $s, t \in \mathcal{G}$, $\alpha, \beta \in \mathbb{R}$:

$$\begin{aligned} \alpha * (s \oplus t) &= \alpha * s \oplus \alpha * t, \quad (11) \\ \alpha * (\beta * s) &= (\alpha\beta) * s, \quad (12) \\ 1 * s &= s, \quad (13) \\ (-1) * s &= s, \quad (14) \\ \sqrt{\alpha^2 + \beta^2} * s &= \alpha * s \oplus \beta * s. \quad (15) \end{aligned}$$

Let us define in $(\mathbb{S}'', \oplus, \mathbb{R}, *)$ two operations as follows:

$$\begin{aligned} \alpha + \beta &= (\alpha^{\widehat{1/2}} \oplus \beta^{\widehat{1/2}})^{\widehat{2}}, \quad (16) \\ \gamma \cdot s &= \gamma * s_{\sigma(\gamma)},\ \gamma \in \mathbb{R}. \quad (17) \end{aligned}$$

It can be easily checked that the space $(\mathbb{S}'', +, \mathbb{R}, \cdot)$ with operations "$+$" and "$\cdot$" defined by (16), (17) is linear. Thus the space $(\mathbb{S}'', \oplus, \mathbb{R}, *)$ can be considered as a linear space with two additional operations, namely $(\mathbb{S}'', +, \oplus, \mathbb{R}, \cdot, *)$.

## 3.2. Inclusion

We next discuss two relations for inclusion of stochastic numbers. The so-called *interval inclusion* (briefly: i-inclusion) is defined for $x_1 = (m_1; s_1),\ x_2 = (m_2; s_2) \in \mathbb{S}$, by:

$$x_1 \subseteq_i x_2 \Longleftrightarrow |m_2 - m_1| \leq s_2 - s_1. \quad (18)$$

We note that addition is inverse i-inclusion isotone, that is: $x_1 + y \subseteq_i x_2 + y$ implies $x_1 \subseteq_i x_2$ [2]. However, it is easy to see that i-inclusion isotonicity does not hold, i. e. $x_1 \subseteq_i x_2$ does not imply $x_1 + y \subseteq_i x_2 + y$. If we want that a two-directional implication

$$x_1 \subseteq x_2 \Longleftrightarrow x_1 + y \subseteq x_2 + y$$

holds in $\mathbb{S}$, then instead of "$\subseteq_i$" we should use the inclusion relation "$\subseteq_s$" between two stochastic numbers defined by

$$x_1 \subseteq_s x_2 \Longleftrightarrow |m_2 - m_1|^2 \leq s_2^{\widehat{2}} - s_1^{\widehat{2}}. \quad (19)$$

Relation (19) is called *stochastic inclusion*, briefly: s-inclusion. In the proper case s-inclusion has been proposed in [1].

**Proposition 1**. Addition and multiplication by scalars are (inverse) inclusion isotone (invariant with respect to s-inclusion).

The proof in the general case ($s$ real) is similar to the one for the proper case ($s$ nonnegative) see [1].

*Remark.* In fact it can be seen that $x_1 \subseteq_s x_2$ if $x_2 = x_1 + y$ for some stochastic number $y = (m, s)$ whose "range" $[m - s, m + s]$ contains 0.

We shall next compare relations (19) and (18). To this end we introduce an end-point presentation.

**End-point presentation.** This presentation may be useful when dealing with confidence intervals. The confidence interval corresponding to the stochastic number $(m; s)$ is $[m - \gamma s, m + \gamma s]$, where $\gamma > 0$ is a chosen number (usually $\gamma \approx 2$). For simplicity in the sequel we assume $\gamma = 1$, which corresponds to usual compact intervals on $\mathbb{R}$.

Recall that the relation between the end-point presentation of an interval $A = [a^-, a^+] \subseteq \mathbb{R}$ and its mid-point/radius presentation $A = (a';\ a'')$ is given by:

$$\begin{aligned} a^- &= a' - a'', \quad a^+ = a' + a''; \\ a' &= (a^- + a^+)/2, \ \ a'' = (a^+ - a^-)/2. \end{aligned}$$

Recall also the relation $a^+ a^- = a'^2 - a''^2$.

The i-inclusion (18) admits a simple end-point presentation, namely for $A \subseteq_i B$ condition $|b' - a'| \le b'' - a''$ is presented in end-point form as $b^- \le a^-$, $a^+ \le b^+$. We next look for an end-point presentation for the s-inclusion (19): $A \subseteq_s B \Longleftrightarrow (b' - a')^2 \le b''^2 - a''^2$.

The condition $(b' - a')^2 \le b''^2 - a''^2$ can be written as $b'^2 - b''^2 + a'^2 + a''^2 \le 2a'b'$. Replacing $b'^2 - b''^2 = b^+ b^-$, $a' = (a^- + a^+)/2$, $a'' = (a^+ - a^-)/2$, etc. we obtain: $2b^+ b^- + {a^+}^2 + {a^-}^2 \le (a^+ + a^-)(b^+ + b^-)$. Thus the end-point condition for s-inclusion obtains the form:

$$A \subseteq_s B \Longleftrightarrow {a^+}^2 + {a^-}^2 + 2b^+ b^- \le (a^+ + a^-)(b^+ + b^-),$$

which can be also written in the form $2(b^+ b^- - a^+ a^-) \le (a^+ + a^-)(b^+ + b^- - (a^+ + a^-))$.

**Proposition 2**. Interval inclusion (18) implies stochastic inclusion (19).

*Proof.* We sketch the proof for proper stochastic numbers. Assume that $A = (a';\ a'')$ is i-included in $B = (b';\ b'')$, $A \subseteq_i B$, which according to (18) means $|b' - a'| \le b'' - a''$. We have to show that (19) holds true. Note first that from (18) we have $0 \le a'' \le b''$. Now from $|b' - a'| \le b'' - a''$ we have $(b' - a')^2 \le (b'' - a'')^2 \le (b'' - a'')(b'' + a'') = b''^2 - a''^2$. □

As a consequence from Proposition 2, stochastic addition is i-inclusion isotone.

### 3.3. Lattice operations

The lattice operations for the i-inclusion are well-known. We next consider the lattice operations for the s-inclusion, sketching the results for proper intervals. The case when one of the stochastic number is s-included in the other one is obvious. Let us determine $\sup(A, B) = C$ for the case when neither $A \subseteq_s B$, nor $B \subseteq_s A$.

Recall that in the case of i-inclusion we have :

$$c'' = |c' - a'| + a'' = |c' - b'| + b''.$$

From $|c' - a'| + a'' = |c' - b'| + b''$ we can compute first $c'$ and then $c'' = |c' - a'| + a''$.

Similarly in the case of s-inclusion we have:

$$c''^2 = a''^2 + (c' - a')^2 = b''^2 + (b' - c')^2.$$

From $a''^2 + (c' - a')^2 = b''^2 + (b' - c')^2$ we compute $c'$:

$$2c' = \frac{b'^2 + b''^2 - (a'^2 + a''^2)}{b' - a'} = \frac{b''^2 - a''^2}{b' - a'} + b' + a'.$$

For $c''^2 = a''^2 + (c' - a')^2$ we obtain:

$$4c''^2 = \frac{(b''^2 - a''^2)^2}{(b' - a')^2} + 2(b''^2 + a''^2) + (b' - a')^2.$$

Let us now determine $\inf(A, B) = D$ for the case when neither $A \subseteq_s B$, nor $B \subseteq_s A$.

Recall first that in the case of i-inclusion we have:

$$d'' = a'' - |d' - a'| = b'' - |d' - b'|.$$

From $a'' - |d' - a'| = b'' - |d' - b'|$ we can compute first $d'$ and then $d'' = a'' - |d' - a'|$.

Similarly in the case of s-inclusion we have:

$$d''^2 = a''^2 - (d' - a')^2 = b''^2 - (b' - d')^2.$$

From $a''^2 - (d' - a')^2 = b''^2 - (b' - d')^2$ we compute $d'$:

$$2d' = \frac{b'^2 - b''^2 - (a'^2 - a''^2)}{b' - a'} = -\frac{b''^2 - a''^2}{b' - a'} + b' + a'.$$

For $d''^2 = a''^2 - (d' - a')^2$ we obtain:

$$4d''^2 = -\frac{(b''^2 - a''^2)^2}{(b' - a')^2} + 2(b''^2 + a''^2) - (b' - a')^2.$$

Clearly, we have $d'' < 0$ for relatively small standard deviations $a'', b''$ and relatively large value of $|b' - a'|$. This means that $\inf(A, B)$ can have a negative standard deviation for proper stochastic numbers. In other words two proper stochastic numbers may not have a proper infimum.

## 4. Application: Lagrange interpolation

The goal of this section is to compare the results obtained with the theory developed in this paper, which is named *continuous stochastic arithmetic (CSA)*, with respective results obtained with the CESTAC method and with interval arithmetic.

As mentioned in the introduction, in the CESTAC method, each stochastic variable is represented by a $N$-tuple of gaussian random values with known mean value $m$ and standard deviation $s$. The method also uses a special arithmetic called *discrete stochastic arithmetic (DSA)*, which acts on the $N$-tuples.

In the scope of granular computing [11], stochastic arithmetic CSA operates on stochastic numbers and is directly derived from operations on independent gaussian random variables. Hence a stochastic number is a granule and CSA is a tool for computing with granules.

With the same point of view of granular computing, in *discrete stochastic arithmetic (DSA)* a granules is composed by a $N$-tuple of samples of the same mathematical result of an arithmetical operator implemented in floating point arithmetic. The samples differ from each other because data are imprecise and because of different rounding. The operator acting on these granules is a floating point operator corresponding to the exact arithmetical operator which is performed $N$ times in a synchronous way with random rounding. Thus the result is also a granule named discrete stochastic number. It has been shown that *DSA* operating on discrete stochastic numbers has many properties (but not all) of real numbers; In particular the notion of stochastic zero has been defined. As explained above, the CADNA library merely implements *DSA*.

To compare the two models, a specific library has been developed which implements both continuous and discrete stochastic arithmetic. The computations are done separately. *CSA* implements the mathematical rules defined above in sections 2–3.

The comparison has been done on the Lagrangian interpolation method. Let $(x_i,\ y_i)$, $i = 0, 1, ..., n$, be a set of $n + 1$ pairs of numbers where all $x_i$ are different. We want to compute the value $p(t)$ of the Lagrangian polynomial at a given point $t$ with the classical formula:

$$p(t) = y_0 l_0(t) + y_1 l_1(t) + \cdots + y_n l_n(t), \qquad (20)$$

wherein

$$l_i(t) = \frac{\prod_{i \neq j}(t - x_j)}{\prod_{i \neq j}(x_i - x_j)}.$$

We consider the situation when the values of $y_i$ are imprecise (contain some errors) and $x_i$ are considered exact. This case is within the scope of our theoretical results where only addition or subtraction between two stochastic numbers and multiplication of a stochastic number by a real number is considered.

For all examples presented below, we take $n = 10$; the exact $x$-values are defined as $x_i = i,\ i = 1, ..., n + 1$, and the imprecise values $y_i$ are around 1. The latter means that in the interval case all intervals $y_i$ have a midpoint 1, whereas in the stochastic case they have a mean value 1.

### 4.1 Interval approach

First, the interval approach is considered to obtain exact bounds for the results assuming that some guaranteed bounds are given for the data $y_i$ in the form of intervals $Y_i$, that is $y_i \in Y_i$.

Then it is well-known that at each $t$

$$p(t) \in P(t) = l_0(t) * Y_0 + l_1(t) * Y_1 + \cdots + l_n(t) * Y_n.$$

The computation of the interval polynomial $P(t)$ has been performed with the Intlab implementation [5] of interval arithmetic. The maximum error on the $Y_i$ value is $ierr = 0.02$. With the case $Y_i = [1 - ierr; 1 + ierr] =$ constant and $x_i = i,\ i = 1, ..., 11$, the upper and lower bounds of $P$ are shown in black on Fig. 1.

*Remark.* This is an example when the use of so-called naive interval arithmetic produces exact (sharp) bounds (Fig. 1). Normally, naive interval arithmetic produces pessimistic bounds. In most cases, such sharp bounds cannot be obtained by naive interval arithmetic and more sophisticate methods should be used.

### 4.2 Experiments with stochastic arithmetic

The computation of (20) is done using the stochastic arithmetic approach (CSA). This approach is based on the gaussian random variable $(m, \sigma)$ with a mean value $m$ and a standard deviation $\sigma$. It is well known that 95% of the samples of a such variable are inside the interval $[m - 2\sigma, m + 2\sigma]$. To compare the results with the interval approach, the $Y_i$ are equal to $(1, 0.01)$.

The computation has been performed with our specific implementation of CSA. A set of $(m_{p(t_i)}, \sigma_{p(t_i)})$ is obtained. The gray lower and upper curves in the figure 1 represent the results of *CSA*, i.e. each point of the lower curve (resp. the upper curve) is equal to $m_{p(t_i)} - 2 * \sigma_{p(t_i)}$ (resp. $m_{p(t_i)} + 2 * \sigma_{p(t_i)}$).

It can be seen that the two curves corresponding to *CSA* are inside the range of those computed with Intlab.

### 4.3 Discrete stochastic arithmetic

The results computed with *DSA* are compared with *CSA* in figures 2, 3, 4, 5. On each figure, the $N$ samples and the lower and upper curve obtained with *CSA* are drawn respectively for $N = 3, 5, 10, 20$. All the figures are composed of two sub-figures. The left sub-figure shows the $N$ curves of the samples. The right sub-figure compares the computed mean value and standard deviation obtained from the $N$-samples and the theoretical mean value and standard deviation obtained with *CSA*.

From these experiments it can be easily seen that if $\overline{p(t_i)}$ denotes the mean value of all the samples obtained with *DSA* for the computation of $p(t_i)$ then: $m_{p(t_i)} - 2\sigma_{p(t_i)} \leq \overline{p(t_i)} \leq m_{p(t_i)} + 2\sigma_{p(t_i)}$. Hence the experiments show clearly that *CSA* is a good model for *DSA* and the CESTAC method.

## 5. Conclusion

Starting from a minimal set of empirically known facts related to stochastic numbers, we formally deduce a number of properties and relations. We investigate the complete set of all stochastic numbers and show that this set possesses nice algebraic properties. We point out to the distinct algebraic nature of the spaces of mean-values and standard deviations. Based on the algebraic properties of the complete set of stochastic numbers we propose a natural relation for inclusion, called stochastic inclusion. A numerical example based on the Lagrange polynomial demonstrates the consistency between the CESTAC method and the presented theory of stochastic numbers.

**Acknowledgments**. The authors want to thank an anonymous referee for his useful comments.

## References

[1] Alt, R., S. Markov, On the Algebraic Properties of Stochastic Arithmetic. Comparison to Interval Arithmetic, In: W. Kraemer and J. Wolff v. Gudenberg (Eds.), Scientific Computing, Validated Numerics, Interval Methods, Kluwer, 2001, 331–342.

[2] Alt, R., J.-L. Lamotte, S. Markov, Abstract structures in stochastic arithmetic, In: B. Bouchon-Meunier, R. R. Yager (Eds.), Proc. 11-th Conference on Information Processing and Management of Uncertainties in Knowledge-based Systems (IPMU 2006), Editions EDK, Paris, 2006, 794–801.

[3] Alt, R., J. Vignes, Validation of Results of Collocation Methods for ODEs with the CADNA Library. *Appl. Numer. Math.* 20 (1996), 1–21.

[4] Chesneaux, J. M., J. Vignes, Les fondements de l'arithmétique stochastique, *C.R. Acad. Sci.*, Paris, Sér. I, Math. 315 (1992), 1435–1440.

[5] INTLAB — INTerval LABoratory Version 5.2., www.ti3.tu-harburg.de/ rump/intlab/

[6] Markov, S., R. Alt, Stochastic arithmetic: Addition and Multiplication by Scalars, *Appl. Numer. Math.* 50 (2004), 475–488.

[7] Markov, S., R. Alt, J.-L. Lamotte, Stochastic Arithmetic: S-spaces and Some Applications, *Numer. Algorithms* 37 (1–4), 275–284, 2004.

[8] Vignes, J., R. Alt, An Efficient Stochastic Method for Round-Off Error Analysis, in: *Accurate Scientific Computations, LNCS 235*, Springer, 1985, 183–205.

[9] Vignes, J., Review on Stochastic Approach to Round-Off Error Analysis and its Applications. *Math. and Comp. in Sim.* 30, 6 (1988), 481–491.

[10] Vignes, J., A Stochastic Arithmetic for Reliable Scientific Computation, *Math. and Comp. in Sim.* 35 (1993), 233–261.

[11] Yao, Y. Y., Granular Computing: basic issues and possible solutions, In: P. P. Wang (Ed.), *Proc. of the 5-th Joint Conference on Information Sciences, Vol. I, Atlantic City, New Jersey, USA, February 27–March 3, 2000*, Association for Intelligent Machinery, 186–189.

[12] `http://www.lip6.fr/cadna`

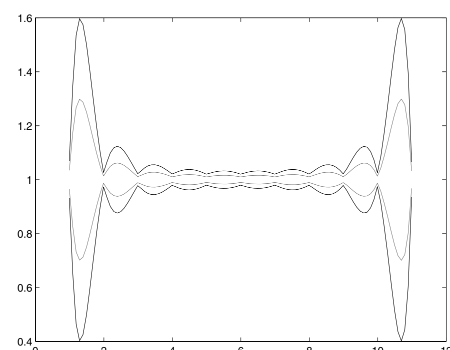


**Figure 1. Lagrangian polynomial obtained with interval arithmetic (black) and CSA (gray)**

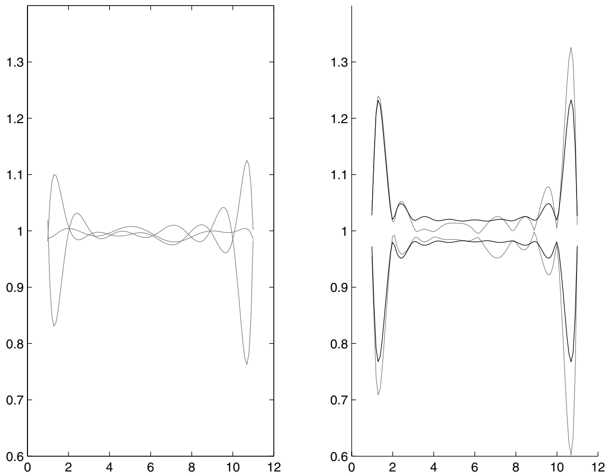


**Figure 2.** *DSA* **3 samples**

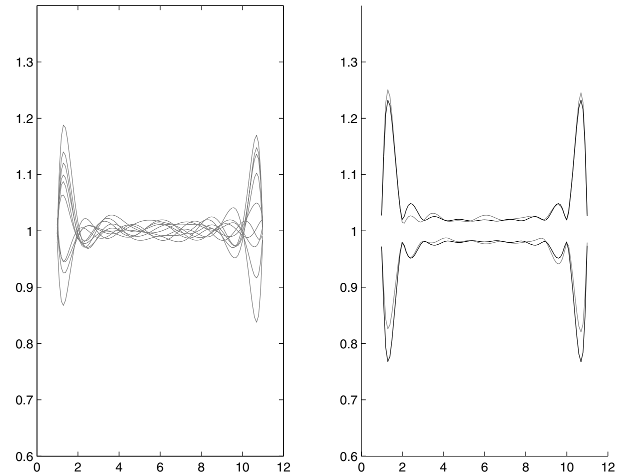


**Figure 4.** *DSA* **10 samples**

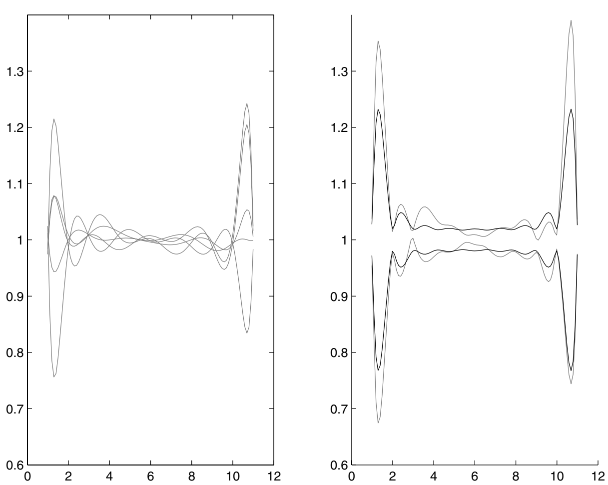


**Figure 3.** *DSA* **5 samples**

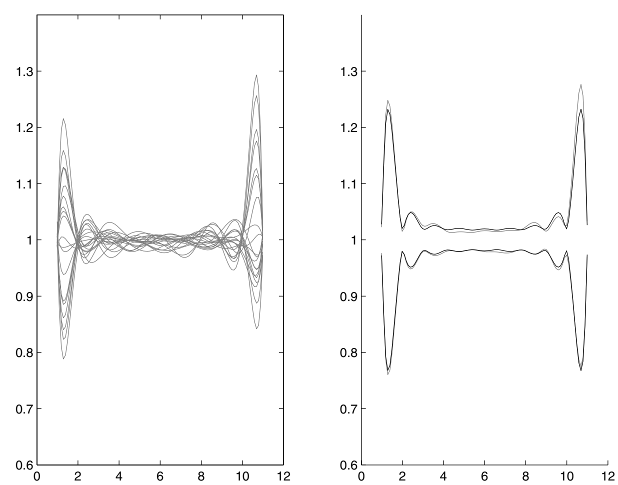


**Figure 5.** *DSA* **20 samples**